Rank functions for modules and categories

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Resumo: Em álgebra existem várias funções que medem uma certa noção de finitude ou tamanho de uma entidade álgebrica. De entre estas funções, as mais úteis tomam geralmente valores nos números reais não negativos e exibem algum tipo de aditividade. Neste texto, focamo-nos num tipo particular de funções, conhecidas como funções de dimensão, definidas em categorias trianguladas. Estas funções inspiram-se na noção clássica de dimensão de um módulo.

Abstract: In algebra one encounters various functions that measure some sort of finiteness or size of an algebraic entity. The most useful of these functions typically take values in the non-negative real numbers and exhibit some form of additivity. In this note, we focus on a particular type of functions, known as rank functions, defined on triangulated categories. These draw their inspiration from the classic notion of rank of a module.

palavras-chave: Função de dimensão; módulo; categoria triangulada.

keywords: Rank function; module; triangulated category.

1 Introduction

The notion of dimension or rank provides a measure of the 'size' of algebraic structures. In the context of modules over a ring, the rank of a module is a fundamental invariant that often signifies the maximal number of linearly independent elements. This concept extends naturally to the study of the so-called rank functions for finitely presented modules.

The study of triangulated categories, which generalise several key structures in algebraic topology, algebraic geometry, and representation theory, has brought forth the demand for analogous 'rank-like' functions. The concept of a rank function on a triangulated category was introduced by Chuang and Lazarev in [CL21], motivated by the work of Cohn, Malcolmson and Schofield on a special type of rank functions for finitely presented modules, called Sylvester rank functions. Work of Crawley-Boevey and Herzog

([Cra94a, Cra94b, Her93]) on additive functions for abelian categories inspired further recent developments on rank functions on triangulated categories ([CGMZ24]).

In this note, we provide an overview of some of the main results on rank functions on triangulated categories. We begin by revisiting rank functions for modules over rings, setting the stage for our exploration of rank functions on triangulated categories. Departing from an analogy with rank functions for modules and following the work of Chuang and Lazarev ([CL21]), we then introduce the notion of a rank function on a triangulated category and present one of the main theorems in [CL21]. Through the use of functorial methods, we demonstrate how rank functions on triangulated categories can be lifted to additive functions on certain associated abelian categories. This allows the translation of known results on the abelian context to the triangulated context and yields new results on rank functions on triangulated categories ([CGMZ24]).

2 Rank functions for modules

To motivate the concept of a rank function on a triangulated category, let us begin with the following open-ended question:

- What is the *rank* of a module?

Before refining the question above, some notation and details are needed. The term 'module' shall indicate a left module over some unitary associative (not necessarily commutative) ring A. Let A-Mod be the category of all A-modules. Recall that an A-module is ***finitely presented*** if it is the cokernel of a morphism between finitely generated free A-modules and denote the full subcategory of finitely presented A-modules by A-mod.

If A is a field (or even a division ring), then a module is simply a vector space, and by rank of a module we usually mean its dimension over A. More generally, the rank of a module is a well-defined concept whenever A is an integral domain. In this broader setting, the rank of a module X is generally understood as the maximal number of A-linear independent elements in X ([DF04]). If A satisfies the invariant basis property, the notion of rank applies to free modules: it is meant as the cardinality of an A-basis for the module. However, if A does not satisfy the invariant basis property, we encounter difficulties.

A general approach to defining a 'rank-like' function involves considering a ring morphism $h : A \to k$ from our underlying ring A to a division ring k.

In this setting, one may define the rank on an A-module X with respect to h as $\mathrm{rk}_h(X) \coloneqq \dim(k \otimes_A X)$, where the right action of A on k is described via the morphism h. This yields an assignment of a non-negative integer to every finitely presented A-module. The assignment rk_h is an instance of an integral Sylvester rank function, as defined below.

Definition 2.1 ([Sch85])**.** A ***rank function*** ρ on A-mod assigns to each object X in A-mod an element $\rho(X) \in \mathbb{R}_{\geq 0}$ such that ρ satisfies the following two axioms:

1. ***Additivity:*** $\rho(X \oplus Y) = \rho(X) + \rho(Y)$ for X and Y in A-mod;
2. ***Triangle inequality:*** $\rho(Z) \leq \rho(Y) \leq \rho(X) + \rho(Z)$ for any right exact sequence of modules in A-mod

$$X \longrightarrow Y \longrightarrow Z \longrightarrow 0.$$

A rank function taking values in $\mathbb{Z}$ is said to be ***integral***. A rank function ρ is called a ***Sylvester rank function*** if $\rho(A) = 1$.

Example 2.2. Let A be a left Artinian ring. In this context, the finitely presented A-modules are exactly those that have finite (Jordan–Hölder) length. Note that the length of an A-module yields an integral rank function. However, this is not a Sylvester rank function in general.

Example 2.3. Let A be an integral domain and let k be its field of fractions. Denote by h the embedding of A into k. One can check that the corresponding integral Sylvester rank function rk_h coincides with the classic notion of rank of a module over an integral domain mentioned previously.

Example 2.4. Let $h : A \to S$ be a ring morphism, where S is a simple left Artinian ring. By the Wedderburn–Artin theorem, S is isomorphic to a matrix ring over a division algebra, i.e. $S \cong M_n(k)$, for some $n \in \mathbb{N}$ and a division ring k. Define

$$\mathrm{rk}_h(X) \coloneqq \frac{\mathrm{length}(S \otimes_A X)}{n}.$$

Note that this assignment generalises the previous definition of rank of X with respect to h and yields a Sylvester rank function taking values in $\mathbb{Q}$.

Much of what we have seen so far hints at a close relationship between rank functions on A-mod and ring morphisms from A to ‘uncomplicated’ rings. This relationship was made precise in the work of Cohn, Malcolmson and Scholfield.

Theorem 2.5 ([Sch85]). *There is a bijection between:*

1. *integral Sylvester rank functions on* A-mod*;*
2. *equivalence classes of ring epimorphisms from* A *to a division ring.*

The inverse correspondence maps a ring epimorphism $h : A \to k$ *to* rk_h*.*

Theorem 2.6 ([Sch85]). *There is a bijection between:*

1. *Sylvester rank functions on* A-mod *taking values on* $\frac{1}{n}\mathbb{Z}$ *with* $n \in \mathbb{Z}_{>0}$*;*
2. *equivalence classes of ring morphisms from* A *to a simple Artinian ring.*

The inverse correspondence maps a ring morphism $h : A \to S$ *to* rk_h*.*

3 Rank functions on triangulated categories

Triangulated categories play a prominent role in algebraic topology, algebraic geometry and representation theory. They provide a framework that generalises several key structures found in these areas, such as derived categories, homotopy categories of stable cofibration categories and stable categories of Frobenius categories. A triangulated category is a category $\mathcal{C}$ equipped with an additional structure subject to certain axioms: an equivalence $\Sigma : \mathcal{C} \to \mathcal{C}$, called a ***translation functor***, together with a class of sequences of three composable morphisms, called ***distinguished triangles***. The precise definition of a triangulated category is lengthy and detailing it here would detract from the main focus of this text, which is to provide a general understanding of rank functions on triangulated categories and their connection to other types of 'measure' functions on different categories. For more details on triangulated categories, we refer, for instance, to [KS06, Nee01].

Suppose henceforth that $\mathcal{C}$ is a skeletally small triangulated category with translation functor $\Sigma : \mathcal{C} \to \mathcal{C}$. The concept of a rank function on a triangulated category was recently introduced by Chuang and Lazarev in [CL21]. Their motivation stemmed from the work by Cohn, Malcolmson and Schofield on Sylvester rank functions ([Coh08, Sch85]).

Definition 3.1 ([CL21]). A ***rank function on objects*** ρ_{ob} of $\mathcal{C}$ assigns to each object X in $\mathcal{C}$ an element $\rho_{ob}(X) \in \mathbb{R}_{\geq 0}$ such that ρ_{ob} satisfies the axioms:

1. ***Additivity:*** $\rho_{ob}(X \oplus Y) = \rho_{ob}(X) + \rho_{ob}(Y)$ for X and Y in $\mathcal{C}$;
2. ***Triangle inequality:*** $\rho_{ob}(Y) \leq \rho_{ob}(X) + \rho_{ob}(Z)$ for any triangle
$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X;$$
3. Σ***-invariance:*** $\rho_{ob}(\Sigma X) = \rho_{ob}(X)$ for every X in $\mathcal{C}$.

Alternatively, rank functions can be defined on morphisms of $\mathcal{C}$.

Definition 3.2 ([CL21])**.** A ***rank function*** ρ on $\mathcal{C}$ assigns to each morphism f in $\mathcal{C}$ an element $\rho(f) \in \mathbb{R}_{\geq 0}$ such that ρ satisfies the following axioms:

1. ***Additivity:*** $\rho(f \oplus g) = \rho(f) + \rho(g)$ for any two morphisms f and g;
2. ***Rank-nullity:*** $\rho(f) + \rho(g) = \rho(\mathbf{1}_Y)$ for any distinguished triangle
$$X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow \Sigma X;$$
3. Σ***-invariance:*** $\rho(\Sigma f) = \rho(f)$ for any morphism f in $\mathcal{C}$.

By [CL21], both definitions above are equivalent by setting $\rho_{ob}(X) \coloneqq \rho(\mathbf{1}_X)$ and
$$\rho(f : X \to Y) \coloneqq \frac{\rho_{ob}(X) + \rho_{ob}(Y) - \rho_{ob}(\operatorname{Cone} f)}{2}.$$
Here, $\operatorname{Cone} f$ denotes the unique object (up to isomorphism) fitting into a distinguished triangle
$$X \xrightarrow{f} Y \longrightarrow \operatorname{Cone} f \longrightarrow \Sigma X.$$

A standard example of a rank function on a triangulated category is the dimension of the total cohomology of an object in the bounded derived category of a finite-dimensional algebra ([CGMZ24]). Mass functions associated with Bridgeland stability conditions provide further interesting examples ([CL21, Ike21]). As the next example shows, it is possible pull back a rank function along a triangulated functor. Note that a ***triangulated functor*** is simply a functor between triangulated categories that preserves the triangulated structure, i.e. it commutes with the respective translation functors and maps distinguished triangles to distinguished triangles.

Example 3.3. Let $h : \mathcal{C} \to \mathcal{D}$ be a triangulated functor an let ρ be a rank function on $\mathcal{D}$. The assignment $\rho_h(f) := \rho(h(f))$ defines a rank function on $\mathcal{C}$.

Developing the analogy between rank functions on triangulated categories and rank functions for modules, and recalling Theorems 2.5 and 2.6, it is natural to wonder which rank functions on $\mathcal{C}$ arise as a pull back of a rank function on an 'uncomplicated' triangulated category $\mathcal{D}$ along a triangulated functor $h : \mathcal{C} \to \mathcal{D}$, as explained in the previous example. To partially answer this question, some further definitions are needed.

Definition 3.4. A rank function ρ on $\mathcal{C}$ is:

1. ***integral*** if $\rho(f) \in \mathbb{Z}$ for every morphism f in $\mathcal{C}$;
2. ***localising*** if it is integral and moreover, if $\rho(f) = 0$ implies that the morphism f factors through some object Z such that $\rho_{ob}(Z) = 0$;
3. ***prime*** if it is integral and $\mathcal{C}$ has a thick generator X such that $\rho(X) = 0$.

In [CL21], Chuang and Lazarev established a connection between prime localising rank functions and certain functors to ***simple triangulated categories***, that is, triangulated categories with an indecomposable generator such that any triangle is a direct sum of split triangles. Compare the next result with Theorems 2.5 and 2.6.

Theorem 3.5 ([CL21])**.** *Suppose that the triangulated category $\mathcal{C}$ has a thick generator. There is a bijection between:*

1. *prime localising rank functions on $\mathcal{C}$;*
2. *thick subcategories $\mathcal{K}$ of $\mathcal{C}$ such that the Verdier quotient $\mathcal{C}/\mathcal{K}$ is a simple triangulated category.*

The inverse correspondence maps a thick subcategory $\mathcal{K}$ of $\mathcal{C}$ to the rank function ρ_h (see Example 3.3), where $h : \mathcal{C} \to \mathcal{C}/\mathcal{K}$ is the Verdier quotient functor and ρ is the unique prime rank function on $\mathcal{C}/\mathcal{K}$.

If $\mathcal{C}$ is the perfect category Per A *of a differential graded algebra A, then 1. and 2. above are also in bijection with:*

3. *equivalence classes of finite homological epimorphisms $A \to S$ with A a simple Artinian differential graded algebra.*

In the subsequent sections we demonstrate how functorial methods can further the analogy between rank functions for modules and rank functions on triangulated categories. This will enhance and generalise the theorem by Chuang and Lazarev discussed above.

4 From rank functions to additive functions

Functorial methods have a history of success in algebra and representation theory. In the following paragraphs, we will explain how these methods can be used to relate rank functions on triangulated categories to similar 'measure' functions on more familiar categories. For this, we will carry on with an analogy between two mathematical realms: additive categories with cokernels and triangulated categories.

An instance of an additive category with cokernels is the category A-mod of finitely presented modules over a ring A. Definition 2.1 remains applicable for an additive category $\mathcal{C}$ with cokernels and there is a well-developed theory of integral rank functions in this broader context ([Cra94a]). The results on rank functions in [Cra94a] are characterised by the following distinctive approach. An abelian category $\mathcal{A}_{\mathcal{C}}$ is constructed from the initial additive category $\mathcal{C}$ and the rank functions on $\mathcal{C}$ are lifted to certain 'measure' functions on $\mathcal{A}_{\mathcal{C}}$, called additive. Working with additive functions on abelian categories is more straightforward, and the results obtained in this context ([Cra94b, Her97]) can be translated back into theorems about rank functions on additive categories with cokernels ([Cra94a]).

In order to apply this strategy to triangulated categories, the first step is to associate an abelian category with a triangulated category $\mathcal{C}$, so that a rank function on $\mathcal{C}$ lifts to an additive function on the associated abelian category. Before explaining how to do this, let us first provide the definition of additive function.

Definition 4.1. An ***additive function*** $\widetilde{\rho}$ on a skeletally small abelian category $\mathcal{A}$ is an assignment $\widetilde{\rho}(X) \in \mathbb{R}_{\geq 0}$ to each object X of $\mathcal{A}$ which satisfies $\widetilde{\rho}(Y) = \widetilde{\rho}(X) + \widetilde{\rho}(Z)$ for any short exact sequence in $\mathcal{A}$

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0.$$

An additive function is said to be ***integral*** if it takes values in $\mathbb{Z}$.

The category Mod-$\mathcal{C}$ of all additive contravariant functors from a skeletally small triangulated category $\mathcal{C}$ to the category of abelian groups is an abelian category with remarkable properties. Let mod-$\mathcal{C}$ be the full subcategory of Mod-$\mathcal{C}$ whose objects are the functors which occur as cokernels of morphisms between ***representable functors***, i.e. between functors naturally isomorphic to $\mathrm{Hom}_{\mathcal{C}}(-,X) : \mathcal{C}^{op} \to \mathrm{Ab}$ for some X in $\mathcal{C}$. The category mod-$\mathcal{C}$ turns out to be a skeletally small abelian category with an equivalence $\Sigma^* : \text{mod-}\mathcal{C} \to \text{mod-}\mathcal{C}$ induced by Σ via $\Sigma^*(F) := F \circ \Sigma^{-1}$. Furthermore, any rank function ρ on the triangulated category $\mathcal{C}$ lifts to an additive function $\widetilde{\rho}$ on mod-$\mathcal{C}$ by setting $\widetilde{\rho}(F) := \rho(f)$, where f is a morphism in $\mathcal{C}$ satisfying $F \cong \mathrm{Im}(\mathrm{Hom}_{\mathcal{C}}(-,f))$. This assignment yields a bijection between rank functions on $\mathcal{C}$ and the additive functions $\widetilde{\rho}$ on mod-$\mathcal{C}$ satisfying $\widetilde{\rho}(\Sigma^* F) = \widetilde{\rho}(F)$.

Theorem 4.2 ([CGMZ24])**.** *Let $\mathcal{C}$ be a skeletally small triangulated category. There is a bijection between:*

1. *Σ-invariant additive functions $\widetilde{\rho}$ on* mod-$\mathcal{C}$*;*
2. *rank functions ρ on $\mathcal{C}$.*

5 Integral rank functions on triangulated categories

By Theorem 4.2, a rank function ρ on a triangulated category $\mathcal{C}$ can be reinterpreted as an additive function $\widetilde{\rho}$ on the category mod-C of finitely presented additive contravariant functors from $\mathcal{C}$ to abelian groups. Through this reinterpretation, the deeply developed theory of additive functions on abelian categories ([Cra94a, Her97]) can be employed to obtain various results about rank functions on triangulated categories. We present some of these results in this section.

The first result is a unique decomposition theorem.

Theorem 5.1 ([CGMZ24])**.** *Every integral rank function on a skeletally small triangulated category $\mathcal{C}$ can be uniquely decomposed as a locally finite sum of irreducible rank functions.*

Here, an ***irreducible*** rank function is a non-zero integral rank function that cannot be further decomposed into a sum of two non-zero integral rank functions. Every prime rank function, for instance, can be shown to be irreducible.

The functorial approach also allows to extend the bijection between localising prime rank functions on $\mathcal{C}$ and thick subcategories of $\mathcal{C}$ with simple

Verdier quotient, established in Theorem 3.5. For this purpose, one needs to introduce a new class of rank functions, called ***exact***, which contains all localising rank functions, and to use the notion of a ***CE-quotient functor***, introduced by Krause in [Kra05]. Instead of functors to simple triangulated categories, one should now consider functors to ***locally finite triangulated categories***, that is, triangulated categories $\mathcal{D}$ such that mod-$\mathcal{D}$ is an abelian category where every object has finite (Jordan–Hölder) length. Instead of prime rank functions, ***basic*** ones need to be used (these are integral rank functions with no repetitions in the irreducible summands appearing in their decomposition).

Theorem 5.2 ([CGMZ24])**.** *Let $\mathcal{C}$ be a skeletally small triangulated category. There is a bijection between:*

1. *exact basic rank functions on $\mathcal{C}$;*
2. *equivalence classes of CE-quotient functors from $\mathcal{C}$ to locally finite triangulated categories.*

Moreover, the underlying rank function is localising if and only if the CE-quotient is equivalent to a Verdier quotient.

In case $\mathcal{C}$ is the perfect derived category of a differential graded algebra A, the following partial generalisation of Theorem 3.5 can be obtained.

Theorem 5.3 ([CGMZ24])**.** *Let A be a cofibrant differential graded algebra. There is a bijection between:*

1. *idempotent basic rank functions on* $\operatorname{Per} A$*;*
2. *equivalence classes of homological epimorphisms $A \to B$ with* $\operatorname{Per} B$ *locally finite.*

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