On certain strongly quasihereditary algebras

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To my parents,
with admiration, gratitude and love.

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Abstract

Given a finite-dimensional algebra A we may associate to it a special endomorphism algebra, R_A, introduced by Auslander. The algebra R_A is a "Schur-like" algebra for A: it contains A as an idempotent subalgebra (up to Morita equivalence) and it is quasihereditary with respect to a particular heredity chain.

The main purpose of this thesis is to describe the quasihereditary structure of R_A which arises from such heredity chain, and to investigate the corresponding Ringel dual of R_A. It turns out that R_A belongs to a certain class of strongly quasihereditary algebras defined axiomatically, which we call ultra strongly quasihereditary algebras. We derive the key properties of ultra strongly quasihereditary algebras, and give examples of other algebras which fit into this setting.

Contents

Introduction

Motivation and aim

Quasihereditary algebras were introduced in [15] by Cline, Parshall and Scott, in order to deal with highest weight categories arising in the representation theory of Lie algebras and algebraic groups. This notion was extensively studied by Dlab and Ringel ([20], [18], [21], [25, Appendix]). Since the introduction of quasihereditary algebras, many classes of algebras arising naturally were shown to be quasihereditary.

A prototype for quasihereditary algebras are the Schur algebras, whose highest weight theory is that of general linear groups. The Schur algebra can be constructed as the endomorphism algebra of a module over the group algebra of a finite symmetric group.

One may ask whether there are analogues of Schur algebras for arbitrary finite-dimensional algebras. That is, given a finite-dimensional algebra A over a field K, we would like to have an A-module whose endomorphism algebra is quasihereditary, so that it has a highest weight theory. Such a module was introduced by Auslander in [5] and he showed that its endomorphism algebra has finite global dimension. Subsequently, Dlab and Ringel proved that this endomorphism algebra is actually a quasihereditary algebra ([18]). The algebra in question is easy to define, namely one can take the endomorphism algebra of the direct sum of all radical powers,

$$G = \bigoplus_{i\colon i \geq 0} A/\operatorname{Rad}^i A$$

(for practical purposes one takes the opposite of the basic version of this algebra). We denote this "Schur-like" endomorphism algebra by R_A and call it the Auslander–Dlab–Ringel algebra (ADR algebra) of A. The original algebra A is then Morita equivalent to $\xi R_A \xi$ for an idempotent ξ in R_A, and this is also analogous to the situation of symmetric groups and Schur algebras.

In this dissertation we study the ADR algebra R_A for an arbitrary finite-dimensional algebra A. The main goal is to understand its highest weight theory.

Short digression on quasihereditary algebras

Quasihereditary algebras are the algebra analogues of a highest weight category. To be precise, the category of modules over a quasihereditary algebra is a highest weight category ([15]).

The most quoted definition of a quasihereditary algebra is as follows. Suppose B is an algebra with simple modules L_i, $i \in \Phi$, where the set Φ has a linear ordering $\sqsubseteq$. Let P_i be the indecomposable projective module with simple quotient isomorphic to L_i. Then the standard module $\Delta(i)$ is defined to be the largest quotient of P_i all of whose composition factors are of the form L_j, with $j \sqsubseteq i$. Dually, the costandard module $\nabla(i)$ is the largest submodule of the injective hull Q_i of L_i all of whose composition factors are of the form L_j, with $j \sqsubseteq i$. These notions make sense in general, but the standard (and costandard) modules depend on the order $(\Phi, \sqsubseteq)$.

Definition. The algebra B is *quasihereditary* with respect to $(\Phi, \sqsubseteq)$ if, for all $i \in \Phi$,

1. L_i occurs only once as a composition factor of $\Delta(i)$;

2. P_i has a filtration whose factors are standard modules.

There is an alternative definition of quasihereditary algebra which only requires Φ to be endowed with a partial order. This other version, which will be used throughout the thesis, involves a technicality and will be discussed in Section 1.4.

Assume that B is a quasihereditary algebra with respect to some order $(\Phi, \sqsubseteq)$. Let $\mathcal{F}(\Delta)$ be the category of all B-modules which have a filtration whose factors are standard modules. The category $\mathcal{F}(\nabla)$ is defined similarly. These two categories are of central interest.

In [50], Ringel has proved that the intersection $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$ contains only finitely many indecomposable modules, and these are labelled by "highest weights": the module $T(i)$ for $i \in \Phi$ has only composition factors of the form L_j, with $j \sqsubseteq i$, and the composition factor L_i occurs once in $T(i)$. In the context of algebraic Lie theory, a module in $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$ is called a "tilting module". We follow the same practice in this thesis. The tilting modules $T(i)$ play a central role in the understanding of decomposition numbers in Lie Theory.

Ringel proved that the endomorphism algebra $\mathcal{R}(B) = \operatorname{End}_B\left(\bigoplus_{i\in\Phi} T(i)\right)$ is again quasihereditary (now with respect to the opposite order, $(\Phi, \sqsubseteq^{op})$). The algebra $\mathcal{R}(B)$ is called the Ringel dual of B, and the Ringel dual of $\mathcal{R}(B)$ is Morita equivalent to B ([50]).

Main contributions

The ADR algebra R_A is quasihereditary – this was proved by Dlab and Ringel in [18] using a notion of quasihereditary algebra different from the one given here. The only additional key result which we found in the literature (and used) on R_A is a theorem due to Smalø, which describes the R_A-module $\operatorname{Hom}_A(G, A/\operatorname{Rad} A)$ ([57]). Otherwise, the results which we now describe are new, but of course we make use of general theory.

We prove that the standard modules over R_A are uniserial and that the category $\mathcal{F}(\Delta)$ is closed under submodules. Ringel calls a quasihereditary algebra B for which $\mathcal{F}(\Delta)$ is closed under submodules a "right strongly quasihereditary algebra", and Dlab and Ringel give various characterisations of such algebras ([19], [51]). The fact that the category $\mathcal{F}(\Delta)$ is closed under submodules seems to suggest that this is a large category. Indeed, the category of A-modules embeds into $\mathcal{F}(\Delta)$: for every A-module M, the R_A-module $\operatorname{Hom}_A(G, M)$ is in $\mathcal{F}(\Delta)$. The tilting modules over the ADR algebra R_A have a very nice structure. We show that each indecomposable tilting module has a unique filtration by costandard modules, and that the indecomposable tilting modules labelled by a maximal weight are injective.

We have placed these results into a more general context by introducing a class of algebras defined axiomatically. It turns out that the class of quasihereditary algebras $(B, \Phi, \sqsubseteq)$ satisfying the axioms below for all $i \in \Phi$

(A1) $\operatorname{Rad}\Delta(i)$ lies in $\mathcal{F}(\Delta)$;

(A2) if $\Delta(i)$ is simple, then the injective hull of $\Delta(i)$ has a filtration by standard modules;

properly contains R_A and encapsulates its main properties. According to Dlab and Ringel ([19]), condition (A1) is satisfied if and only if the category of modules with a Δ-filtration is closed under submodules (i.e. if and only if B is a right strongly quasihereditary algebra). This prompts the following definition.

Definition. A quasihereditary algebra $(B, \Phi, \sqsubseteq)$ is *right ultra strongly quasihereditary* (RUSQ, for short) if it satisfies axioms (A1) and (A2).

We prove several properties for RUSQ algebras, and for their Ringel duals. We show that the standard modules are uniserial and that the simple modules can be labelled in a natural way by pairs (i, j) so that $\Delta(i, j)$ has radical $\Delta(i, j+1)$ for $1 \leq j < l_i$ and $\Delta(i, l_i)$ is simple. The next result summarises some of our findings.

Theorem. *Let B be a RUSQ algebra. The injective hull Q_{i,l_i} of the simple B-module with label (i, l_i) lies in $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$ (that is, Q_{i,l_i} is a tilting module). Moreover, the chain of inclusions*

$$0 \subset T(i, l_i) \subset \cdots \subset T(i, j) \subset \cdots \subset T(i, 1) = Q_{i,l_i},$$

where $T(i, j)$ is the tilting module corresponding to the label (i, j), is the unique filtration of Q_{i,l_i} by costandard modules. For $1 \leq j < l_i$, the injective hull $Q_{i,j}$ of the simple module with label (i, j) is isomorphic to $Q_{i,l_i}/T(i, j+1)$.

In addition, the Ringel dual $\mathcal{R}(B)$ of B is such that $\mathcal{R}(B)^{op}$ is still a RUSQ algebra.

The ADR algebra R_A is described in detail, by quiver and relations, when A is the group algebra of the symmetric group Σ_p and the underlying field K has characteristic p. The algebra obtained is, naturally, different from the Schur algebra $S_K(p, p)$.

We also determine the projective covers of a large class of modules over the ADR algebra. For an A-module M, the projective cover of the R_A-module $\operatorname{Hom}_A(G, M)$ is the image of a map ε through the functor $\operatorname{Hom}_A(G, -)$. The morphism ε is a special kind of map: it is a right minimal $\operatorname{add} G$-approximation (see [7]). We describe the $\operatorname{add} G$-approximations of rigid A-modules. Note that a module is rigid if its descending radical series is equal to its ascending radical series.

Theorem A. *Let M be a rigid A-module with radical length m. Then the projective cover of M as an $(A/\operatorname{Rad}^m A)$-module is a right minimal* $\operatorname{add} G$*-approximation of M.*

This last result is then used to provide a counterexample to a claim by Auslander, Platzeck and Todorov in [6] about the projective resolutions of modules over the ADR algebra, for which no arguments were given.

Theorem A is also a key ingredient in the proof of the following central result of this thesis.

Theorem B. *Suppose that all indecomposable projective and injective A-modules have the same radical length, and are also rigid. Then the Ringel dual $\mathcal{R}(R_A)$ of R_A is isomorphic to the algebra $(R_{A^{op}})^{op}$.*

In order to establish this theorem, we prove some auxiliary results of independent interest. Namely, we fully describe the filtrations by standard modules of the tilting R_A-modules in the case where all the projective indecomposable A-modules are rigid and have the same radical length.

Our work on the ADR algebra is also compared with existing literature. For example, we explain why one should not expect the ADR algebra to be isomorphic to its own Ringel dual, contrary to what happens often for quasihereditary algebras arising from semisimple Lie algebras and algebraic groups. Furthermore, we look into the representation type of the categories $\mathcal{F}(\Delta)$ and $\mathcal{F}(\nabla)$, in connection with the representation type of A.

The ADR algebras, the quasihereditary algebras described by Iyama in [39] (used to prove the finiteness of the representation dimension), and the cluster-tilted algebras studied in [32], [31] and [40] are all examples of strongly quasihereditary algebras. We conclude this thesis by showing that all these algebras are instances of a generic construction which produces strongly quasihereditary endomorphism algebras.

Outline of the chapters

Chapter 1 is designed to provide the reader with the necessary background. It introduces basic concepts of representation theory, the language of preradicals, and the fundamentals of quasihereditary algebras.

In Chapter 2, we begin the study of the quasihereditary structure of R_A that stems from the heredity chain in [18]. We show that the ADR algebra is a RUSQ algebra, as stated previously. We then proceed to derive the basic properties of the tilting modules and of the injective modules over arbitrary RUSQ algebras (Theorem 2.5.8, Proposition 2.5.11), and we investigate the Ringel dual of such algebras (Theorem 2.6.1). We also compute R_A when A is a certain Brauer tree algebra.

Chapter 3 is devoted to explore the connection between the Ringel dual of R_A and the ADR algebra of A^{op}. We show that Ringel dual of R_A is isomorphic to $(R_{A^{op}})^{op}$ under certain regularity conditions for A (Theorem B). In order to prove this, we study the Δ-filtrations of modules over RUSQ algebras (Corollary 3.2.3, Lemma 3.2.15, Theorem 3.4.5), and the projective covers of the R_A-modules $\mathrm{Hom}_A(G, M)$ for M a rigid A-module (Theorem A). Along the way, we give a counterexample to a remark by Auslander–Platzeck–Todorov in [6].

In Chapter 4 we answer further natural questions about the ADR algebra and relate our work to relevant literature (Section 4.2). We also lay the basis for some of the problems investigated in Chapter 5 (Section 4.3).

In Chapter 5, we describe a general way of constructing strongly quasihereditary endomorphism algebras (Theorem 5.2.8). We show that the ADR algebras, the algebras constructed by Iyama in [39], and the cluster-tilted algebras studied in [32], [31] and [40] all fit into our setting. Moreover, we provide sufficient conditions for which the algebras obtained by our construction are ultra strongly quasihereditary algebras (Theorem 5.4.5).

A list of notation and an index may be found at the end of the thesis.

Chapter 1

Background and notation

1.1 Overview of the chapter

This chapter contains notation, remarks and standard results which will be used constantly throughout this thesis. Section 1.2 discusses the fundamentals of algebras, Artin algebras and their representations. Section 1.3 contains material on preradicals and Section 1.4 deals with basic material on quasihereditary algebras.

The theory developed in this thesis is mainly aimed at finite-dimensional algebras over a field. However, for most of this dissertation, we work with slightly more general classes of rings. The reason for this is that the added generality considerably widens the applicability of the theory, but requires minimal extra effort.

The notion of quasihereditary algebra is ubiquitous in this thesis. Quasihereditary algebras originated from the study of the category $\mathcal{O}$, the category of rational representations of an algebraic group, and also from the study of Schur algebras. There are many different ways of defining a quasihereditary algebra, but they turn out to be equivalent. The most often used definition of a quasihereditary algebra involves labelling the simple modules by a linearly ordered set. We define quasihereditary algebras using partially ordered sets instead, as these are better suited for our purposes.

The language and formalism of preradicals will also be particularly convenient to us, specially in Chapters 3 and 5, and more generally when dealing with quasihereditary algebras. The concept of a preradical is closely tied to that of a torsion theory.

The results in this chapter are classic and part of the "mathematical folklore". However, proofs are provided where no precise references are known.

1.2 Algebras and Artin algebras

We start by giving the definition of an algebra and of an Artin algebra, and by recalling some of their basic properties. Our chosen definition of algebra is quite broad. It covers not only the class of Artin algebras, but also infinite-dimensional algebras over a field, namely infinite-dimensional path algebras and bound quiver algebras. The notion of an Artin algebra is a generalisation of that of a finite-dimensional algebra over a field. Artin algebras have a well-developed representation theory, which includes a consistent left-right symmetry and duality theory.

1.2.1 Definitions and basic properties

Definition 1.2.1 ([8, II.1]). Let C be a commutative artinian ring with unit. A *C-algebra* A is a ring with unit, together with a (unit preserving) ring morphism $\phi : C \longrightarrow A$ whose image is in the centre of A. For $c \in C$ and $a \in A$, write ca for $\phi(c)a$. A C-algebra A is said to be an *Artin algebra* if A is finitely generated as a module over C.

For most of this dissertation we work with Artin algebras, having in mind applications to finite-dimensional algebras. The only exception to this is in Chapter 5. There, our initial setting is general enough to incorporate infinite-dimensional algebras over a field, and in particular the preprojective algebra.

Denote the category of left modules over a C-algebra A by $\operatorname{Mod} A$. Every A-module is a C-module in a canonical way. Let $\operatorname{mod} A$ be the category of all left A-modules which are finitely generated over C. Evidently, the modules in $\operatorname{mod} A$ are finitely generated A-modules.

If A is an Artin algebra, then the modules in $\operatorname{mod} A$ are exactly the finitely generated A-modules. In particular, A lies in $\operatorname{mod} A$, and A is both a left and a right artinian ring (see [8, II.1] for the latter assertion). For a field K and a K-algebra A (possibly infinite-dimensional over K), $\operatorname{mod} A$ corresponds to the category of finite dimensional A-modules.

We give a brief list of basic properties of the category $\operatorname{mod} A$, and refer to the textbooks [2] and [8] for further details.

Proposition 1.2.2. *Let A be a C-algebra. The following holds:*

1. *the modules in* $\operatorname{mod} A$ *have finite length (i.e. they have a composition series);*

2. *every monic (resp. epic) endomorphism of a module in* $\operatorname{mod} A$ *is an isomorphism;*

3. *every module in* $\operatorname{mod} A$ *has a decomposition as a direct sum of a finite number of indecomposable modules in* $\operatorname{mod} A$*, which is unique up to isomorphism and permutation of the summands;*

4. *the indecomposable modules in* $\operatorname{mod} A$ *are exactly the modules in* $\operatorname{mod} A$ *with local endomorphism ring;*

5. *the category* $\operatorname{mod} A$ *is a Krull-Schmidt abelian subcategory of the abelian category* $\operatorname{Mod} A$*;*

6. *for* M *and* N *in* $\operatorname{mod} A$*,* $\operatorname{Hom}_A(M, N)$ *is a finitely generated* C*-module,*

7. *for* M *and* N *in* $\operatorname{mod} A$*, the endomorphism algebra* $\Gamma = \operatorname{End}_A(M)^{op}$ *is an Artin* C*-algebra and* $\operatorname{Hom}_A(M, N)$ *lies in* $\operatorname{mod} \Gamma$*;*

Proof. Let M be a module in $\operatorname{mod} A$. Since C is artinian, then, by Hopkins' Theorem ([2, Theorem 15.20]), C is also noetherian. Since M is finitely generated over C, then it is both artinian and noetherian as a C-module (see Propositions 10.18 and 10.19 in [2]). Consequently, M satisfies both the ascending and the descending chain condition as an A-module (since chains of A-modules are in particular chains of C-modules). By Proposition 11.1 in [2], M has finite length as an A-module. This proves the statement of part 1.

Part 2 corresponds to Proposition 1.4 in [8, I]. Parts 3, 4 and 5 basically coincide with the statement of the Krull-Schmidt Theorem (see Theorem 12.9, Lemma 12.8, Corollary 11.8, Lemma 12.8 and Corollary 11.2 in [2]).

For the proof of parts 6 and 7 we follow the reasoning in the proof of Proposition 1.1 in [8]. It is easy to check that $\operatorname{Hom}_A(M, N)$ is a C-submodule of $\operatorname{Hom}_C(M, N)$. We show that $\operatorname{Hom}_C(M, N)$ is finitely generated as a C-module. By Proposition 10.19 in [2], it will then follow that $\operatorname{Hom}_A(M, N)$ is a finitely generated C-module as well. Since M is finitely generated over C, there is an epimorphism of C-modules $C^n \longrightarrow M$, for some $n \in \mathbb{Z}_{\geq 0}$. This gives rise to a monomorphism of C-modules $\operatorname{Hom}_C(M, N) \longrightarrow \operatorname{Hom}_C(C^n, N)$, where $\operatorname{Hom}_C(C^n, N) \cong N^n$. The C-module N^n is finitely generated. Proposition 10.19 in [2] implies that $\operatorname{Hom}_C(M, N)$ is also a finitely generated C-module.

For part 7, note that $\operatorname{End}_A(M)$ is a C-subalgebra of $\operatorname{End}_C(M)$. By part 6, these two algebras are finitely generated over C, hence they are Artin algebras. The abelian group $\mathrm{Hom}_A(M,N)$ is a Γ-module, and it is finitely generated over C by part 6. This concludes the proof of the proposition. □

Throughout this dissertation the letters A and B will be used to denote C-algebras and Artin algebras, with the letter B mainly reserved for quasihereditary algebras. The letter K represents a field, and we will often specialise to finite-dimensional algebras over a field whenever it is convenient. The term *module* will designate a left module over some algebra. Virtually all A-modules studied in this thesis lie in $\mathrm{mod}\, A$, and, by default, all modules are assumed to be in $\mathrm{mod}\, A$. The notation $[M : L]$ will be used for the multiplicity of a simple module L in the composition series of a module M in $\mathrm{mod}\, A$.

We say that a class of modules (or a single module) Θ *generates* a module M if M is the image of an epic whose domain is a direct sum of modules in Θ. Dually, we say that Θ *cogenerates* M if there is a monic from M to a direct product of modules in Θ. For a module M in $\mathrm{mod}\, A$, these direct sums and direct products of modules in Θ can be assumed to be finite, that is, they can be reduced to finite direct sums (see Propositions 10.10, 10.1 and 10.2 in [2]).

Given M in $\mathrm{mod}\, A$, denote the *radical* of M by $\mathrm{Rad}\, M$, that is, let $\mathrm{Rad}\, M$ be the smallest submodule of M such that $M/\mathrm{Rad}\, M$ is semisimple. We call the semisimple module $M/\mathrm{Rad}\, M$ the *top* of M, and denote it by $\mathrm{Top}\, M$. Finally, let $\mathrm{Soc}\, M$ denote the *socle* of M, i.e. let $\mathrm{Soc}\, M$ be the largest semisimple submodule of M. The operators $\mathrm{Rad}^i(-)$ and $\mathrm{Soc}_i(-)$ are defined recursively by the identities $\mathrm{Rad}^i M = \mathrm{Rad}(\mathrm{Rad}^{i-1} M)$ and $\mathrm{Soc}_i M/\mathrm{Soc}_{i-1} M = \mathrm{Soc}(M/\mathrm{Soc}_{i-1} M)$.

1.2.1.1 Properties of Artin algebras

Suppose now that A is an Artin C-algebra. Let J be the direct sum of all the injective indecomposable C-modules, with one representative from each isomorphism class. Denote by D the functor

$$\mathrm{Hom}_C(-,J) : \mathrm{mod}\, A \longrightarrow (\mathrm{mod}(A^{op}))^{op}.$$

The functor D induces an equivalence of categories, or, in other words, it induces a duality between $\mathrm{mod}\, A$ and $\mathrm{mod}(A^{op})$ (see [8, II.3]). We call D the *standard duality* for Artin algebras. If A is a finite-dimensional algebra over a field K, this functor reduces to the vector space duality $\mathrm{Hom}_K(-,K)$.

Every module in $\mathrm{mod}\, A$ is generated by some projective in $\mathrm{mod}\, A$. The existence of the duality D implies that every module in $\mathrm{mod}\, A$ is also cogenerated by some injective in $\operatorname{mod} A$. Let $L_1, \ldots, L_n$ be a complete set of nonisomorphic simple A-modules. We shall denote the projective cover of L_i (resp. the injective hull of L_i) by P_i (resp. by Q_i).

Proposition 1.2.3 ([8, III, Proposition 1.15]). *Let A be an Artin algebra. Using the previous notation, the following numbers are the same:*

1. $\dim_{\operatorname{End}_A(L_j)} \operatorname{Ext}^1_A(L_i, L_j)$;
2. $[\operatorname{Rad} P_i / \operatorname{Rad}^2 P_i : L_j]$;
3. *the multiplicity of P_j as a summand of the projective cover of* $\operatorname{Rad} P_i$.

The following numbers are the same:

1. $\dim_{\operatorname{End}_A(L_i)^{op}} \operatorname{Ext}^1_A(L_i, L_j)$;
2. $[\operatorname{Soc}_2 Q_j / \operatorname{Soc} Q_j : L_i]$;
3. *the multiplicity of Q_i as a summand of the injective hull of $Q_j / \operatorname{Soc} Q_j$.*

1.2.2 Approximations

The concept of (minimal) approximation was introduced by Auslander and Smalø ([9]), and independently by Enochs ([27]). Approximations arise naturally in representation theory, in particular in tilting theory.

Let A be an algebra. A map $f : M \longrightarrow N$ in $\operatorname{mod} A$ is called a *right minimal morphism* if every endomorphism $g : M \longrightarrow M$ satisfying $f = f \circ g$ is an automorphism. Dually, $f : M \longrightarrow N$ in $\operatorname{mod} A$ is *left minimal* if an endomorphism $g : N \longrightarrow N$ is an automorphism whenever $f = g \circ f$. See [9, Section 1] or [7, Proposition 1.1] for the key properties of minimal morphisms. Note, in particular, that nonzero morphisms with indecomposable domain (resp. codomain) are always right minimal (resp. left minimal).

Let now $\mathcal{X}$ be a class of modules in $\operatorname{mod} A$. A morphism $f : X \longrightarrow M$ in $\operatorname{mod} A$, with X in $\mathcal{X}$, is said to be a *right $\mathcal{X}$-approximation* of M if

$$\operatorname{Hom}_A(X', X) \xrightarrow{\operatorname{Hom}_A(X', f)} \operatorname{Hom}_A(X', M) \longrightarrow 0$$

is exact for all X' in $\mathcal{X}$. Finally, say that a map is a *right minimal $\mathcal{X}$-approximation* if it is both a right $\mathcal{X}$-approximation and a right minimal morphism. It follows directly from the definition that right minimal $\mathcal{X}$-approximations are unique up to isomorphism (if they exist). That is, given right minimal $\mathcal{X}$-approximations f_1 and f_2 of a module M in $\operatorname{mod} A$ there is an isomorphism g satisfying $f_1 = f_2 \circ g$.

Similarly, a map $f : M \longrightarrow X$, with X in $\mathcal{X}$, is a *left $\mathcal{X}$-approximation* of M if every morphism $g : M \longrightarrow X'$, with X' in $\mathcal{X}$, factors through f. *Left minimal $\mathcal{X}$-approximations* are defined as naturally expected, and they are also unique up to isomorphism (in case they exist).

1.2.3 Projectivisation

Let M be in $\operatorname{mod} A$. Passing from the C-algebra A to the Artin C-algebra $\Gamma := \operatorname{End}_A(M)^{op}$ turns questions about the module M into questions about projective modules. This correspondence is established via the functor

$$\operatorname{Hom}_A(M, -) : \operatorname{mod} A \longrightarrow \operatorname{mod} \Gamma,$$

which is left exact and commutes with finite direct sums.

Let Θ be a class of modules in $\operatorname{mod} A$. Denote the *additive closure of* Θ by $\operatorname{add} \Theta$, that is, let $\operatorname{add} \Theta$ be the full subcategory of $\operatorname{mod} A$ whose modules are isomorphic to summands of finite direct sums of modules in Θ. The additive closure of a single module M is denoted by $\operatorname{add} M$.

Proposition 1.2.4. *The functor* $\operatorname{Hom}_A(M, -)$ *has the following properties:*

1. *for* X *in* $\operatorname{add} M$ *and* N *in* $\operatorname{mod} A$*, the functor* $\operatorname{Hom}_A(M, -)$ *induces an isomorphism of* C*-modules*

$$\operatorname{Hom}_A(X, N) \longrightarrow \operatorname{Hom}_\Gamma(\operatorname{Hom}_A(M, X), \operatorname{Hom}_A(M, N));$$

2. *if* X *is in* $\operatorname{add} M$*, then* $\operatorname{Hom}_A(M, X)$ *lies in* $\operatorname{proj} \Gamma$*, the full subcategory of projective modules in* $\operatorname{mod} \Gamma$*;*

3. *the restriction of the functor* $\operatorname{Hom}_A(M, -)$ *to* $\operatorname{add} M$ *induces an equivalence of categories*

$$\operatorname{Hom}_A(M, -) : \operatorname{add} M \longrightarrow \operatorname{proj} \Gamma;$$

4. *for* N *in* $\operatorname{mod} A$*, the maps in part 1 take right* $\operatorname{add} M$*-approximations of* N *to epimorphisms, and vice versa; this bijective correspondence takes right minimal* $\operatorname{add} M$*-approximations to projective covers; in particular, right (minimal)* $\operatorname{add} M$*-approximations do exist.*

Proof. Parts 1 to 3 correspond to the statement of Proposition 2.1, in Chapter II of [8] – although our setting is slightly more general, the proof in [8] works without requiring any changes. Part 4 follows from the definition of right approximation and right minimal morphism, from the additivity of the Hom functors and from part 1. □

1.3 Preradicals

Preradicals generalise the classic notions of radical and socle of a module. The results and definitions stated in this section are elementary and most of the proofs may be found in [11], [14, Chapter 2] and [59, Chapter VI]. Throughout this section, $\mathcal{C}$ generically represents the category $\operatorname{Mod} B$ or the category $\operatorname{mod} B$, where B is some C-algebra. The methods employed can be easily adapted to complete and cocomplete abelian categories – see [59, Chapter VI].

1.3.1 Definition and first properties

Definition 1.3.1. A *preradical* τ in $\mathcal{C}$ is a subfunctor of the identity functor $1_{\mathcal{C}}$, i.e., τ assigns to each module M in $\mathcal{C}$ a submodule $\tau(M)$, such that each morphism $f : M \longrightarrow N$ in $\mathcal{C}$ induces a morphism $\tau(f) : \tau(M) \longrightarrow \tau(N)$ given by restriction.

Obviously, every preradical in $\operatorname{Mod} B$ can be restricted to a preradical in $\operatorname{mod} B$.

A submodule N of a module M in $\operatorname{Mod} B$ is called a *characteristic submodule* of M if $f(N) \subseteq N$, for every f in $\operatorname{End}_B(M)$. By definition, it is clear that the module $\tau(M)$ is a characteristic submodule of M, for every preradical τ in $\mathcal{C}$ and for every M in $\mathcal{C}$. It is also evident that every preradical is an additive functor which preserves monics.

To each preradical τ in $\mathcal{C}$ we may associate the functor

$$1/\tau : \mathcal{C} \longrightarrow \mathcal{C}, \tag{1.1}$$

which maps M in $\mathcal{C}$ to $M/\tau(M)$. Note that the functor $1/\tau$ preserves epics.

Example 1.3.2. For any class Θ of modules in $\operatorname{Mod} B$, the operators defined by

$$\operatorname{Tr}(\Theta, M) := \sum_{f :\, f \in \operatorname{Hom}_B(U,M),\, U \in \Theta} \operatorname{Im} f,$$
$$\operatorname{Rej}(M, \Theta) := \bigcap_{f :\, f \in \operatorname{Hom}_B(M,U),\, U \in \Theta} \operatorname{Ker} f,$$

for M in $\mathcal{C}$, are preradicals in $\mathcal{C}$ (see [2, §8] for further details on these functors). Note that the module $\operatorname{Tr}(\Theta, M)$, called the *trace of* Θ *in* M, is the largest submodule of M generated by Θ. Symmetrically, $\operatorname{Rej}(M, \Theta)$, the *reject of* Θ *in* M, is the submodule N of M such that M/N largest factor module of M cogenerated by Θ. If ε is a complete set of simples in $\operatorname{mod} B$, then $\operatorname{Tr}(\varepsilon, M) = \operatorname{Soc} M$ and $\operatorname{Rej}(M, \varepsilon) = \operatorname{Rad} M$ for M in $\operatorname{mod} B$.

The statements below are easy consequences of the definition of preradical.

Lemma 1.3.3. *Let τ be a preradical in $\mathcal{C}$. Suppose that N and M are in $\mathcal{C}$, with $N \subseteq M$, and let $(M_i)_{i\in I}$ be a finite family of modules in $\mathcal{C}$. The following holds:*

1. *if $\tau(N) = N$, then $N \subseteq \tau(M)$;*
2. *if $\tau(M/N) = 0$, then $\tau(M) \subseteq N$;*
3. *$\tau\left(\bigoplus_{i\in I} M_i\right) = \bigoplus_{i\in I} \tau(M_i)$;*
4. *$\tau(B)$ is an ideal of B (assuming that B is in $\mathcal{C}$);*
5. *$\tau(B)\, M \subseteq \tau(M)$ (assuming that B is in $\mathcal{C}$).*

Proof. Parts 1 and 2 are straightforward and part 3 follows from the fact that preradicals are additive functors.

Suppose that B is in $\mathcal{C}$. For part 4, note that $\tau(B)$ is a left ideal of B. Let $b \in B$ and consider the morphism $r_b : B \longrightarrow B$ given by right multiplication by b. The module $\tau(B)$ is invariant under the action of r_b, i.e. we have $r_b(\tau(B)) \subseteq \tau(B)$. Hence $\tau(B)$ is a right ideal of B, and part 4 holds.

Let now m be in M and consider the map $r_m : B \longrightarrow M$ given by $r_m(b) = bm$, $b \in B$. We have that

$$\tau(B)\, m = r_m(\tau(B)) \subseteq \tau(M)\,.$$

This proves part 5. □

To a preradical τ in $\mathcal{C}$ we associate the classes

$$\mathbb{T}_\tau := \{N \in \mathcal{C} :\, \tau(N) = N\},$$
$$\mathbb{F}_\tau := \{N \in \mathcal{C} :\, \tau(N) = 0\}.$$

The class $\mathbb{T}_\tau$ is called the *pretorsion class of* τ and $\mathbb{F}_\tau$ is called the *pretorsion free class of* τ.

1.3.2 Hereditary and cohereditary preradicals

We shall now look at preradicals which satisfy specific properties.

Definition 1.3.4. A preradical τ in $\mathcal{C}$ is called *idempotent* if $\tau \circ \tau = \tau$. Symmetrically, we say that τ is a *radical* if $\tau \circ (1/\tau) = 0$.

Example 1.3.5. Note that the functor $\mathrm{Soc}\,(-)$ is an idempotent preradical in $\mathrm{mod}\, B$. Similarly, the functor $\mathrm{Rad}\,(-)$ is a radical in $\mathrm{mod}\, B$.

Remark 1.3.6. More generally, the functor $\mathrm{Tr}\,(\Theta, -)$ is an idempotent preradical in $\mathcal{C}$ for every class of modules Θ in $\mathrm{Mod}\, B$. Furthermore, if τ is an idempotent preradical in $\mathcal{C}$, then $\tau = \mathrm{Tr}\,(\mathbb{T}_\tau, -)$. These assertions are easy to check. For a reference consult, for instance, [14, Proposition 6.8].

Similarly, a preradical τ is a radical in $\mathcal{C}$ if and only if it can be defined as $\mathrm{Rej}\,(-, \Theta)$, for some class of modules Θ in $\mathcal{C}$. Namely, if τ is a radical, then $\tau = \mathrm{Rej}\,(-, \mathbb{F}_\tau)$.

Note that $\tau\,(N) \subseteq N \cap \tau\,(M)$ for M and N in $\mathcal{C}$ satisfying $N \subseteq M$. Moreover, by applying τ to the canonical epic $M \longrightarrow M/N$, we conclude that $(\tau\,(M) + N)\,/N \subseteq \tau\,(M/N)$.

Definition 1.3.7. A preradical τ in $\mathcal{C}$ is *hereditary* if $\tau\,(N) = N \cap \tau\,(M)$, for all M and N in $\mathcal{C}$ such that $N \subseteq M$. Dually, a preradical τ in $\mathcal{C}$ is said to be *cohereditary* if $(\tau\,(M) + N)\,/N = \tau\,(M/N)$ for $N \subseteq M$, M and N in $\mathcal{C}$.

Example 1.3.8. The functors $\mathrm{Soc}\,(-)$ and $\mathrm{Rad}\,(-)$ are the typical examples of a hereditary preradical and of a cohereditary preradical in $\mathrm{mod}\, B$, respectively.

Lemma 1.3.9. *Let τ be a preradical in $\mathcal{C}$. The following statements are equivalent:*

1. *τ is hereditary;*

2. *τ is a left exact functor;*

3. *the functor $1/\tau$ preserves monics.*

Moreover, any hereditary preradical is idempotent.

Proof. If τ is hereditary then $\tau\,(\tau\,(M)) = \tau\,(M) \cap \tau\,(M) = \tau\,(M)$, i.e. τ is idempotent.

We prove that assertion 1 implies assertion 2. For this, consider a short exact sequence in $\mathcal{C}$,

$$0 \longrightarrow N \stackrel{f}{\longrightarrow} M \stackrel{g}{\longrightarrow} N' \longrightarrow 0 \quad .$$

Recall that τ preserves monics. Hence, it suffices to prove that $\operatorname{Ker}\tau(g) \subseteq \operatorname{Im}\tau(f)$. Note that

$$\operatorname{Ker}\tau(g) = \operatorname{Ker} g \cap \tau(M) = \tau(\operatorname{Ker} g) = \tau(\operatorname{Im} f),$$

where the second equality follows from the fact that τ is hereditary. Let $f^{|}$ be the map obtained from f by restricting its codomain to $\operatorname{Im} f$. Since τ is a functor, it preserves isomorphisms, so $\tau(f^{|})$ is an isomorphism. In particular, $\operatorname{Im}\tau(f^{|}) = \tau(\operatorname{Im} f)$. Furthermore, observe that $\operatorname{Im}\tau(f^{|}) = \operatorname{Im}\tau(f)$. This proves the implication.

To see that assertion 3 follows from assertion 2, consider a monic $f : N \longrightarrow M$. Because τ is left exact, the short exact sequence

$$0 \longrightarrow N \xrightarrow{f} M \xrightarrow{\pi} \operatorname{Coker} f \longrightarrow 0$$

is mapped to the exact sequence

$$0 \longrightarrow \tau(N) \xrightarrow{\tau(f)} \tau(M) \xrightarrow{\tau(\pi)} \tau(\operatorname{Coker} f) \quad .$$

Let $n \in N$ be such that $f(n) \in \tau(M)$ (so $(\tau(\pi))(f(n))$ is defined). We claim that $n \in \tau(N)$. We have that

$$(\tau(\pi))(f(n)) = \pi(f(n)) = 0,$$

hence $f(n)$ lies in $\operatorname{Ker}\tau(\pi) = \operatorname{Im}\tau(f) = f(\tau(N))$. Because f is monic, n must be in $\tau(N)$. This proves that the map $1/\tau(f) : N/\tau(N) \longrightarrow M/\tau(M)$ is monic.

For the last implication, let N be a submodule of M. By assumption, the functor $1/\tau$ sends the inclusion map to the monomorphism $\iota : N/\tau(N) \longrightarrow M/\tau(M)$. We claim that $N \cap \tau(M) \subseteq \tau(N)$. Let $n \in N \cap \tau(M)$. We have that $n + \tau(N)$ lies in $\operatorname{Ker}\iota = 0$, so n lies in $\tau(N)$. □

Remark 1.3.10. There is a result "dual" to Lemma 1.3.9 for cohereditary predicals in $\mathcal{C}$: a preradical τ is cohereditary if and only if $1/\tau$ is a right exact functor, if and only if the functor τ preserves epics. Furthermore, cohereditary preradicals are radicals.

Example 1.3.11. Let $\mathcal{P}$ be a class of projective modules in $\operatorname{Mod} B$. Given $N \subseteq M$ in $\mathcal{C}$, any map $f : P \longrightarrow M/N$, with P in $\mathcal{P}$, lifts to a map $f' : P \longrightarrow M$. So $\operatorname{Im} f = (\operatorname{Im} f' + N)/N$, thus

$$\operatorname{Tr}(\mathcal{P}, M/N) \subseteq (\operatorname{Tr}(\mathcal{P}, M) + N)/N,$$

i.e. $\operatorname{Tr}(\mathcal{P}, -)$ is a cohereditary preradical in $\mathcal{C}$. By Remark 1.3.6, the preradical $\operatorname{Tr}(\mathcal{P}, -)$ is also idempotent. Similarly, for a class $\mathcal{Q}$ of injective modules in $\operatorname{Mod} B$, the functor $\operatorname{Rej}(-, \mathcal{Q})$ is a hereditary preradical which is also a radical.

Remark 1.3.12. Let B be an Artin algebra and let $\mathcal{P}$ be a class of projective modules in $\operatorname{mod} B$. Observe that $\operatorname{Tr}(\mathcal{P}, M) = \operatorname{Tr}(\mathcal{P}, B)\, M$ for M in $\operatorname{mod} B$. Moreover, $\operatorname{Tr}(\mathcal{P}, B)$ is an idempotent ideal of B. To prove these claims, consider an epic $f : B^n \longrightarrow M$, $n \in \mathbb{Z}_{>0}$ (such a map always exists). According to the previous example, $\tau := \operatorname{Tr}(\mathcal{P}, -)$ is a cohereditary preradical. Thus, the map $\tau(f)$ is still an epic. So

$$\begin{aligned} \tau(M) &= f(\tau(B^n)) = f((\tau(B))^n) \\ &= f(\tau(B)(B^n)) = \tau(B)\, f(B^n) \\ &= \tau(B)\, M. \end{aligned}$$

In particular, we get that $(\tau(B))^2 = \tau(\tau(B))$. Finally, note that $\tau(\tau(B)) = \tau(B)$, as $\operatorname{Tr}(\mathcal{P}, -)$ is an idempotent preradical.

It is possible to construct hereditary (and cohereditary) preradicals out of special classes of modules.

Definition 1.3.13. A class $\blacktriangle$ of modules in $\operatorname{mod} B$ is *hereditary* if every submodule of a module in $\operatorname{add} \blacktriangle$ is generated by $\blacktriangle$. Dually, a class $\blacktriangledown$ of modules in $\operatorname{mod} B$ is *cohereditary* if every factor module of a module in $\operatorname{add} \blacktriangledown$ is cogenerated by $\blacktriangledown$.

Lemma 1.3.14. *If $\blacktriangle$ is a hereditary class of modules in $\operatorname{mod} B$ then $\operatorname{Tr}(\blacktriangle, -)$ is a hereditary preradical in $\operatorname{mod} B$.*

Proof. Consider $N \subseteq M$, with M and N in $\operatorname{mod} B$. The module $\operatorname{Tr}(\blacktriangle, M)$ is generated by some module M' which is a (finite) direct sum of modules in $\blacktriangle$. Consider the pullback square

$$\begin{array}{ccc} M'' & \dashrightarrow\!\!\!\!\rightarrow & N \cap \operatorname{Tr}(\blacktriangle, M) \\ \downarrow & & \downarrow \\ M' & \twoheadrightarrow & \operatorname{Tr}(\blacktriangle, M) \end{array} .$$

As $\blacktriangle$ is a hereditary class, then M'' is generated by $\blacktriangle$. Hence $N \cap \operatorname{Tr}(\blacktriangle, M)$ is generated by $\blacktriangle$ as well. Since $\operatorname{Tr}(\blacktriangle, N)$ is the largest submodule of N generated by $\blacktriangle$, we must have $N \cap \operatorname{Tr}(\blacktriangle, M) \subseteq \operatorname{Tr}(\blacktriangle, N)$. □

Remark 1.3.15. There is a result dual to Lemma 1.3.14 for cohereditary classes of modules in $\operatorname{mod} B$.

Let τ and υ be preradicals in $\mathcal{C}$. We write $\tau \le \upsilon$ if τ is a subfunctor of υ. Consider the functor $\tau \circ \upsilon$ – this is still a preradical. Note that $\tau \circ \upsilon \le \upsilon$ and $\tau \circ \upsilon \le \tau$. For M in $\mathcal{C}$ define $\tau \bullet \upsilon(M)$ as the submodule of M containing $\upsilon(M)$, satisfying

$$\tau(M/\upsilon(M)) = \tau \bullet \upsilon(M) / \upsilon(M).$$

We check that $\tau \bullet \upsilon$ is a predadical in $\mathcal{C}$.

Lemma 1.3.16. *Let τ and υ be preradicals. Then $\tau \bullet \upsilon$ is a preradical.*

Proof. Let $f : M \longrightarrow N$ be a map in $\mathcal{C}$ and consider $m \in \tau \bullet \upsilon(M)$. The element $f(m) + \upsilon(N) \in N/\upsilon(N)$ is the image of $m + \upsilon(M)$ through the map $\tau \circ (1/\upsilon)(f)$. Hence $f(m) + \upsilon(N)$ lies in $\tau \bullet \upsilon(N)/\upsilon(N)$, so $f(m)$ belongs to $\tau \bullet \upsilon(N)$. □

By construction, $\upsilon \leq \tau \bullet \upsilon$, and it is easy to check that $\tau \leq \tau \bullet \upsilon$. Moreover, τ is a radical if and only if $\tau \bullet \tau = \tau$.

By the characterisation of hereditary radicals given in Lemma 1.3.9, it follows that $\tau \circ \upsilon$ is hereditary in $\mathcal{C}$ if τ and υ are both hereditary in $\mathcal{C}$. Similarly, if τ and υ are both cohereditary then $\tau \circ \upsilon$ is cohereditary. We also have that $\tau \bullet \upsilon$ is hereditary (resp. cohereditary), whenever τ and υ are both hereditary (resp. both cohereditary) – this is because the functor $1/(\tau \bullet \upsilon)$ is naturally isomorphic to $(1/\tau) \circ (1/\upsilon)$.

Similarly to the composition of preradicals, the operation $\bullet$ is associative.

Lemma 1.3.17. *The operation $\bullet$ is associative.*

Proof. Let τ, υ and μ be preradicals and let M be in $\mathcal{C}$. Consider the canonical isomorphism

$$(M/\mu(M))/(\upsilon \bullet \mu(M)/\mu(M)) \longrightarrow M/\upsilon \bullet \mu(M).$$

The preradical τ maps this morphism to the restriction isomorphism between

$$\begin{aligned}\tau((M/\mu(M))/(\upsilon \bullet \mu(M)/\mu(M))) &= \tau((M/\mu(M))/\upsilon(M/\mu(M))) \\ &= \tau \bullet \upsilon(M/\mu(M))/\upsilon(M/\mu(M)) \\ &= ((\tau \bullet \upsilon) \bullet \mu(M)/\mu(M)) \\ &\qquad /(\upsilon \bullet \mu(M)/\mu(M)).\end{aligned}$$

and

$$\tau(M/\upsilon \bullet \mu(M)) = \tau \bullet (\upsilon \bullet \mu)(M)/\upsilon \bullet \mu(M).$$

This implies that the operation $\bullet$ is associative. □

In general we have

$$\tau \bullet (\upsilon \circ \mu) \neq (\tau \bullet \upsilon) \circ \mu, \ \tau \circ (\upsilon \bullet \mu) \neq (\tau \circ \upsilon) \bullet \mu.$$

1.3.3 Filtrations arising from preradicals

Given a preradical τ in $\mathcal{C}$, let τ^0 be the identity functor in $\mathcal{C}$ and let τ_0 be the zero preradical. For $m \in \mathbb{Z}_{>0}$, define $\tau^m := \tau \circ \tau^{m-1}$ and $\tau_m := \tau \bullet \tau_{m-1}$. The next lemma summarises the properties of these preradicals.

Lemma 1.3.18. *Let τ be a preradical in $\mathcal{C}$.*

1. *For every $m \geq 1$, $\tau^m \leq \tau^{m-1}$ and $\tau_{m-1} \leq \tau_m$.*
2. *Given $m, m' \geq 0$, then $\tau^m \circ \tau^{m'} = \tau^{m+m'}$.*
3. *Given $m, m' \geq 0$, then $\tau_m \bullet \tau_{m'} = \tau_{m+m'}$.*
4. *For every M in $\operatorname{mod} B$ there is $m \geq 0$ such that $\tau^m (M) = \tau^{m+1} (M)$.*
5. *For every M in $\operatorname{mod} B$ there is $m \geq 0$ such that $\tau_m (M) = \tau_{m+1} (M)$.*
6. *If τ is a radical then $\tau_m = \tau$, for every $m \geq 1$.*
7. *If τ is idempotent then $\tau^m = \tau$, for every $m \geq 1$.*

The preradicals τ^m and τ_m, $m \in \mathbb{Z}_{\geq 0}$, give rise to special filtrations of modules in $\mathcal{C}$.

Lemma 1.3.19. *Let τ be a preradical in $\mathcal{C}$. Suppose that $\tau (M) \neq 0$ for every nonzero module M in $\mathcal{C}$. Given M in $\operatorname{mod} B$, there is a unique integer $l^{(\tau,\bullet)} (M) = n \geq 0$ such that $\tau_n (M) = M$, and $\tau_{m-1} (M) \subset \tau_m (M)$ for every m satisfying $1 \leq m \leq n$. Moreover, for $m \leq l^{(\tau,\bullet)} (M)$, we have*

$$l^{(\tau,\bullet)} (M/\tau_m (M)) = n - m.$$

Proof. It is clear that there is an integer n satisfying $\tau_n (M) = M$ and $\tau_{m-1} (M) \subset \tau_m (M)$ for every m, $1 \leq m \leq n$. Notice that

$$\tau_{n-m} (M/\tau_m (M)) = (\tau_{n-m} \bullet \tau_m (M)) /\tau_m (M) = M/\tau_m (M) ,$$

for $1 \leq m \leq n$. If $n - m \geq 1$, we have

$$\tau_{n-m-1} (M/\tau_m (M)) = \tau_{n-1} (M) /\tau_m (M) \subset M/\tau_m (M) ,$$

which concludes the proof of the lemma. □

Remark 1.3.20. There is a similar result for preradicals τ in $\mathcal{C}$ satisfying $\tau(M) \neq M$ for every nonzero M: given M in $\operatorname{mod} B$, there is a unique integer $l^{(\tau,\circ)}(M) = n \geq 0$ such that $\tau^n(M) = 0$, and $\tau^m(M) \subset \tau^{m-1}(M)$ for every m satisfying $1 \leq m \leq n$; further, if $m \leq l^{(\tau,\circ)}(M)$, we have $l^{(\tau,\circ)}(\tau_m(M)) = n - m$.

Lemma 1.3.21. *Let τ be a hereditary preradical in $\mathcal{C}$. Then τ_m is also a hereditary preradical, and $\tau_m \circ \tau_{m'} = \tau_{\min\{m,m'\}}$ for every $m, m' \geq 0$. Furthermore, if $\tau(M) \neq 0$ for every $M \neq 0$, the following holds for N and M in* $\operatorname{mod} B$*:*

1. *if $N \subseteq M$ then $l^{(\tau,\bullet)}(N) \leq l^{(\tau,\bullet)}(M)$;*
2. *if $N \subseteq \tau_m(M)$, then $l^{(\tau,\bullet)}(N) \leq m$;*
3. *if $m \leq l^{(\tau,\bullet)}(M)$, then $\tau_m(M)$ is the largest submodule N of M such that $l^{(\tau,\bullet)}(N) = m$.*

Proof. Let τ be a hereditary preradical. We have seen that applying the binary operation $\bullet$ to hereditary preradicals yields a hereditary preradical, so τ_m is hereditary whenever τ is. Let now M be in $\operatorname{mod} B$. As τ_m is hereditary, we have that

$$\tau_m(\tau_{m'}(M)) = \tau_{m'}(M) \cap \tau_m(M) = \tau_{\min\{m,m'\}}(M).$$

Suppose further that $\tau(M) \neq 0$ for every $M \neq 0$. For part 1, let $l^{(\tau,\bullet)}(M) = n$. Then

$$\tau_n(N) = N \cap \tau_n(M) = N \cap M = N.$$

For part 2, note that

$$\tau_m(N) = N \cap \tau_m(\tau_m(M)) = N \cap \tau_m(M) = N.$$

Part 3 is now straightforward. □

As expected, there is a symmetric result for cohereditary preradicals τ satisfying $\tau(M) \neq M$ for every $M \neq 0$.

1.4 Quasihereditary algebras

In this section we gather some results on quasihereditary algebras and on the Ringel duality. We discuss two (equivalent) definitions of quasihereditary algebra which are particularly well suited for our purposes. Both notions follow the module theoretical perspective of [21], and use partial orders instead of linear orders, contrary to the most quoted definitions of quasihereditary algebra. The main references for this section are [20], [18], [21], [25, Appendix] and [6]. The letter B shall denote an Artin algebra.

1.4.1 Standard and costandard modules

Given an Artin algebra B, we may label the isomorphism classes of simple B-modules by the elements of a finite set Φ. Denote the simple B-modules by L_i, $i \in \Phi$, and use the notation P_i (resp. Q_i) for the projective cover (resp. the injective hull) of L_i.

For a subset Φ' of Φ define

$$\mathcal{P}_{\Phi'} := \{P_i : i \in \Phi'\},$$
$$\mathcal{Q}_{\Phi'} := \{Q_i : i \in \Phi'\}.$$

Remark 1.4.1. Let M be in $\operatorname{mod} B$. The module $\operatorname{Tr}(\mathcal{P}_{\Phi'}, M)$ is the largest submodule of M which is generated by projectives in $\mathcal{P}_{\Phi'}$. Recall that $\operatorname{Tr}(\mathcal{P}_{\Phi'}, -)$ is a cohereditary preradical in $\operatorname{mod} B$ (see Example 1.3.11), hence it is a radical (according to Remark 1.3.10). Therefore, by Remark 1.3.6, we have

$$\operatorname{Tr}(\mathcal{P}_{\Phi'}, -) = \operatorname{Rej}\left(-, \mathbb{F}_{\operatorname{Tr}(\mathcal{P}_{\Phi'},-)}\right),$$

where

$$\begin{aligned}\mathbb{F}_{\operatorname{Tr}(\mathcal{P}_{\Phi'},-)} &= \{N \in \operatorname{mod} B : \operatorname{Tr}(\mathcal{P}_{\Phi'}, N) = 0\} \\ &= \{N \in \operatorname{mod} B : \text{all composition factors of } N \text{ are of type } L_i,\ i \notin \Phi'\}.\end{aligned}$$

Thus, there is an alternative description of $\operatorname{Tr}(\mathcal{P}_{\Phi'}, -)$. For M in $\operatorname{mod} B$, $\operatorname{Tr}(\mathcal{P}_{\Phi'}, M)$ is the submodule N of M such that M/N is the largest quotient of M all of whose composition factors are of the form L_i, $i \notin \Phi'$.

Similarly, $\operatorname{Rej}(M, \mathcal{Q}_{\Phi'})$ may also be described in two different ways. It is the submodule N of M such that M/N is the largest quotient of M cogenerated by injectives in $\mathcal{Q}_{\Phi'}$. The module $\operatorname{Rej}(M, \mathcal{Q}_{\Phi'})$ is also the largest submodule of M all of whose composition factors are of the form L_i, $i \notin \Phi'$.

Suppose from now onwards that the labelling set Φ is endowed with a partial order $\sqsubseteq$. Given $i \in \Phi$, let $\Delta(i)$ be the largest quotient of P_i all of whose composition factors are of the form L_j, with $j \sqsubseteq i$, that is, define

$$\Delta(i) = \Delta_{(\Phi,\sqsubseteq)}(i) := P_i / \operatorname{Tr}\left(\mathcal{P}_{\Phi_{\not\sqsubseteq i}}, P_i\right) = P_i / \operatorname{Tr}\left(\bigoplus_{j:\, j \not\sqsubseteq i} P_j, P_i\right), \tag{1.2}$$

where

$$\Phi_{\not\sqsubseteq i} := \{j \in \Phi : j \not\sqsubseteq i\}.$$

The modules $\Delta(i)$ are called *standard modules* and we set

$$\Delta := \{\Delta(i) : i \in \Phi\}.$$

Dually, let $\nabla(i)$, $i \in \Phi$, be the largest submodule of Q_i all of whose composition factors are of the form L_j, with $j \sqsubseteq i$, i.e.,

$$\nabla(i) = \nabla_{(\Phi,\sqsubseteq)}(i) := \operatorname{Rej}\left(Q_i, \mathcal{Q}_{\Phi_{\not\sqsubseteq i}}\right) = \operatorname{Rej}\left(Q_i, \bigoplus_{j:\, j \not\sqsubseteq i} Q_j\right). \tag{1.3}$$

We set

$$\nabla := \{\nabla(i) : i \in \Phi\}$$

and call the elements in ∇ the *costandard modules*.

The sets Δ and ∇ depend, in an essential way, on the indexing poset $(\Phi, \sqsubseteq)$. In general, the standard and costandard modules change when we refine the poset $(\Phi, \sqsubseteq)$. In order to prevent this from happening we consider adapted orders in the sense of the definition below.

Definition 1.4.2 ([21])**.** An indexing poset $(\Phi, \sqsubseteq)$ for B is called *adapted* provided that the following condition holds: for every M in $\operatorname{mod} B$ with $\operatorname{Top} M \cong L_i$ and $\operatorname{Soc} M \cong L_j$, where i and j are incomparable elements in Φ, there is $k \in \Phi$ with $k \sqsupset i$ or $k \sqsupset j$, such that L_k is a composition factor of M.

Remark 1.4.3. Note that an indexing set Φ for B which is endowed with a linear order is always adapted.

Lemma 1.4.4 ([21])**.** *Let $(\Phi, \sqsubseteq)$ be an adapted order for the set of simple B-modules and suppose that $(\Phi, \sqsubseteq')$ is a refinement of $(\Phi, \sqsubseteq)$. Then $\Delta_{(\Phi,\sqsubseteq)}(i) = \Delta_{(\Phi,\sqsubseteq')}(i)$ and $\nabla_{(\Phi,\sqsubseteq)}(i) = \nabla_{(\Phi,\sqsubseteq')}(i)$ for every $i \in \Phi$, and the poset $(\Phi, \sqsubseteq')$ is still adapted.*

The standard and the costandard modules have very useful homological properties.

Lemma 1.4.5 ([21])**.** *Let $(\Phi, \sqsubseteq)$ be an adapted order for the Artin algebra B. Take $i, j \in \Phi$. The following holds:*

1. *if* $\operatorname{Ext}^1_B(\Delta(i), \Delta(j)) \neq 0$, *then* $i \sqsubset j$;

2. *if* $\operatorname{Ext}^1_B(\nabla(i), \nabla(j)) \neq 0$, *then* $i \sqsupset j$;

3. $\operatorname{Ext}^1_B(\Delta(i), \nabla(j)) = 0$.

Suppose $|\Phi| = n$ and let $(\Phi, \sqsubseteq')$ be a *total extension* of $(\Phi, \sqsubseteq)$, i.e. suppose that $(\Phi, \sqsubseteq')$ is a refinement of the poset $(\Phi, \sqsubseteq)$ to a linear order. Given such a refinement, the elements in Φ may be relabeled as $k_1, \ldots, k_n$, where

$$k_1 \sqsubset' \cdots \sqsubset' k_i \sqsubset' \cdots \sqsubset' k_n.$$

Let M be in $\operatorname{mod} B$. We define its *trace filtration* with respect to the linear order $(\Phi, \sqsubseteq')$ by

$$0 = M_{(n)}^{(\Phi,\sqsubseteq')} \subseteq \cdots \subseteq M_{(i)}^{(\Phi,\sqsubseteq')} \subseteq \cdots \subseteq M_{(0)}^{(\Phi,\subseteq')} = M,$$

with $M_{(i)}^{(\Phi,\sqsubseteq')} = \operatorname{Tr}\left(\{P_{k_j} : j > i\}, M\right) = \operatorname{Tr}(\mathcal{P}_{\Phi_{\not\sqsubseteq' k_i}}, M)$.

In a similar way, the *reject filtration* of M associated to the total extension $(\Phi, \sqsubseteq')$ of $(\Phi, \sqsubseteq)$ is defined by

$$0 = M_{[0]}^{(\Phi,\sqsubseteq')} \subseteq \cdots \subseteq M_{[i]}^{(\Phi,\sqsubseteq')} \subseteq \cdots \subseteq M_{[n]}^{(\Phi,\sqsubseteq')} = M,$$

with $M_{[i]}^{(\Phi,\sqsubseteq')} = \operatorname{Rej}\left(M, \{Q_{k_j} : j > i\}\right) = \operatorname{Rej}(M, \mathcal{Q}_{\Phi_{\not\sqsubseteq' k_i}})$.

We shall now turn our attention to the modules which have a trace filtration whose factors lie in $\operatorname{add}\Delta$ and to the modules possessing a reject filtration whose factors belong to $\operatorname{add}\nabla$.

Given a class of modules Θ, let $\mathcal{F}(\Theta)$ be the full subcategory of $\operatorname{mod} B$ consisting of all modules that have a Θ-*filtration*, that is, a filtration whose factors lie in Θ (up to isomorphism). The categories $\mathcal{F}(\Delta)$ and $\mathcal{F}(\nabla)$ are of central importance. We call the elements of $\mathcal{F}(\Delta)$ (resp. of $\mathcal{F}(\nabla)$) Δ-*good* modules (resp. ∇-*good* modules). The next proposition relates trace filtrations (resp. reject filtrations) with the category $\mathcal{F}(\Delta)$ (resp. $\mathcal{F}(\nabla)$).

Proposition 1.4.6 ([25, Appendix])**.** *Let M be in $\operatorname{mod} B$ and suppose that $(\Phi, \sqsubseteq)$ is an adapted poset for B.*

1. *The module M belongs to $\mathcal{F}(\Delta)$ if and only if, for any total extension $(\Phi, \sqsubseteq')$ of $(\Phi, \sqsubseteq)$, the factor $M_{(i-1)}^{(\Phi,\sqsubseteq')}/M_{(i)}^{(\Phi,\sqsubseteq')}$ in the trace filtration of M associated to $(\Phi, \sqsubseteq')$ is a direct sum of copies of $\Delta(k_i)$, $i = 1, \ldots, n$.*

2. *The module M belongs to $\mathcal{F}(\Delta)$ if and only if, for any total extension $(\Phi, \sqsubseteq')$ of $(\Phi, \sqsubseteq)$, the factor $M_{[i]}^{(\Phi,\sqsubseteq')}/M_{[i-1]}^{(\Phi,\sqsubseteq')}$ in the reject filtration of M associated to $(\Phi, \sqsubseteq')$ is a direct sum of copies of $\nabla(k_i)$, $i = 1, \ldots, n$.*

Remark 1.4.7. It follows from the proof of this result that Δ- and ∇-filtrations are essentially unique (*à la* Jordan–Hölder). For M in $\mathcal{F}(\Delta)$ and $i \in \Phi$, denote the multiplicity of the standard module $\Delta(i)$ in a Δ-filtration of M by $(M : \Delta(i))$. The number $(M : \nabla(i))$ is defined analogously for M in $\mathcal{F}(\nabla)$.

It is immediate that the classes $\mathcal{F}(\Delta)$ and $\mathcal{F}(\nabla)$ are closed under extensions. Moreover, part 3 of Lemma 1.4.5 implies (by induction) that

$$\operatorname{Ext}^1_B(\mathcal{F}(\Delta), \mathcal{F}(\nabla)) = 0.$$

The next result summarises the main properties of the classes $\mathcal{F}(\Delta)$ and $\mathcal{F}(\nabla)$.

Lemma 1.4.8 ([21], [46, Theorem 3.2])**.** *Suppose that $(\Phi, \sqsubseteq)$ is an adapted order for B. Then:*

1. *both $\mathcal{F}(\Delta)$ and $\mathcal{F}(\nabla)$ are closed under extensions;*

2. $\operatorname{Ext}^1_B(\mathcal{F}(\Delta), \mathcal{F}(\nabla)) = 0$*;*

3. *any short exact sequence whose terms are in $\mathcal{F}(\Delta)$ (resp. $\mathcal{F}(\nabla)$) and whose middle term is a direct sum of standard modules (resp. direct sum of costandard modules) splits;*

4. *if $M_1 \oplus M_2$ is in $\mathcal{F}(\Delta)$ (resp. in $\mathcal{F}(\nabla)$) then both M_1 and M_2 belong to $\mathcal{F}(\Delta)$ (resp. $\mathcal{F}(\nabla)$);*

5. *the class $\mathcal{F}(\Delta)$ is closed under kernels of epics;*

6. *the class $\mathcal{F}(\nabla)$ is closed under cokernels of monics.*

1.4.2 Definition of a quasihereditary algebra

Definition 1.4.9. Let B be an Artin algebra with an indexing poset $(\Phi, \sqsubseteq)$. We say that B is *quasihereditary* with respect to $(\Phi, \sqsubseteq)$ (and write $(B, \Phi, \sqsubseteq)$) if the following conditions hold:

1. $(\Phi, \sqsubseteq)$ is adapted to B;

2. for every $i \in \Phi$, $[\Delta(i) : L_i] = 1$;

3. for every $i \in \Phi$, P_i is in $\mathcal{F}(\Delta)$.

Remark 1.4.10. Note that condition 2 above is equivalent to the conditions:

2'. for every $i \in \Phi$, $\operatorname{End}_B(\Delta(i))$ is a division ring;

2". for every $i \in \Phi$, $[\nabla(i) : L_i] = 1$;

2"'. for every $i \in \Phi$, $\operatorname{End}_B(\nabla(i))$ is a division ring.

Moreover, condition 3 in the previous definition may be replaced by the condition:

3'. for every $i \in \Phi$, $Q_i \in \mathcal{F}(\nabla)$,

provided that conditions 1 and 2 hold.

As mentioned before, the numbers $(M : \Delta(i))$ and $(N : \nabla(i))$ are independent of a choice of a Δ-filtration and of a ∇-filtration for $M \in \mathcal{F}(\Delta)$ and $N \in \mathcal{F}(\nabla)$, respectively. Quasihereditary algebras satisfy a Brauer–Humphreys type of reciprocity, which reduces to the identities $(P_i : \Delta(j)) = [\nabla(j) : L_i]$ and $(Q_i : \nabla(j)) = [\Delta(j) : L_i]$ when the field K is algebraically closed ([21, Lemma 2.5]). More generally, we have the following well-known result.

Lemma 1.4.11. *Let $(B, \Phi, \sqsubseteq)$ be a quasihereditary algebra. Let M and N be B-modules, with $M \in \mathcal{F}(\Delta)$ and $N \in \mathcal{F}(\nabla)$. Then, for $i \in \Phi$,*

$$(M : \Delta(i)) = \dim_{\mathrm{End}_B(\nabla(i))} \mathrm{Hom}_B(M, \nabla(i)),$$
$$(N : \nabla(i)) = \dim_{\mathrm{End}_B(\Delta(i))^{op}} \mathrm{Hom}_B(\Delta(i), N).$$

Proof. Because $\mathrm{End}_B(\Delta(i))^{op}$ is a division ring, it is possible to compute the dimension of $\mathrm{Hom}_B(\Delta(i), N)$ over $\mathrm{End}_B(\Delta(i))^{op}$. Note that $\mathrm{Hom}_B(\Delta(i), \nabla(j))$ is nonzero if and only if $j = i$. Moreover, since $\mathrm{Ext}^1_B(\mathcal{F}(\Delta), \mathcal{F}(\nabla)) = 0$ (see Lemma 1.4.8), the functor $\mathrm{Hom}_B(\Delta(i), -)$ preserves exactness when applied to short exact sequences with modules in $\mathcal{F}(\nabla)$. So, by induction on the number of costandard modules appearing on a ∇-filtration of N, one concludes that $(N : \nabla(i))$ is given by

$$\dim_{\mathrm{End}_B(\Delta(i))^{op}} \mathrm{Hom}_B(\Delta(i), N) / \dim_{\mathrm{End}_B(\Delta(i))^{op}} \mathrm{Hom}_B(\Delta(i), \nabla(i)).$$

We now claim that $\mathrm{End}_B(\Delta(i))$ and $\mathrm{Hom}_B(\Delta(i), \nabla(i))$ are isomorphic as modules over $\mathrm{End}_B(\Delta(i))^{op}$. In order to see this, apply the functor $\mathrm{Hom}_B(\Delta(i), -)$ to the exact sequences

$$0 \longrightarrow \mathrm{Rad}\,\Delta(i) \longrightarrow \Delta(i) \longrightarrow L_i \longrightarrow 0$$

$$0 \longrightarrow L_i \longrightarrow \nabla(i) \longrightarrow \nabla(i) / \mathrm{Soc}\,\nabla(i) \longrightarrow 0 \quad .$$

On the one hand, we deduce that the modules $\mathrm{End}_B(\Delta(i))$ and $\mathrm{Hom}_B(\Delta(i), L_i)$ are isomorphic (note that $\mathrm{Ext}^1_B(\Delta(i), \mathrm{Rad}\,\Delta(i)) = 0$ since $\mathrm{Ext}^1_B(\Delta(i), L_j) = 0$ for every $j \sqsubseteq i$). On the other hand, it follows that $\mathrm{Hom}_B(\Delta(i), L_i) \cong \mathrm{Hom}_B(\Delta(i), \nabla(i))$. Thus the modules $\mathrm{End}_B(\Delta(i))$ and $\mathrm{Hom}_B(\Delta(i), \nabla(i))$ are isomorphic over the division algebra $\mathrm{End}_B(\Delta(i))^{op}$. So $\dim_{\mathrm{End}_B(\Delta(i))^{op}} \mathrm{Hom}_B(\Delta(i), \nabla(i)) = 1$, and we get the desired identity. The proof of the identity involving M is similar. □

It is easy to conclude that the property of being quasihereditary is closed under Morita equivalence. Moreover, every quasihereditary algebra has finite global dimension ([55]). Finally, note that if B is a quasihereditary algebra with respect to the poset $(\Phi, \sqsubseteq)$ then B^{op} is quasihereditary with respect to the same poset.

Proposition 1.4.12 gives an alternative characterisation of a quasihereditary algebra.

Proposition 1.4.12. *Let B an Artin algebra, and suppose that $(\Phi, \sqsubseteq)$ is an indexing poset for the simple B-modules. Then B is quasihereditary with respect to $(\Phi, \sqsubseteq)$ if and only if the following conditions hold for every $i \in \Phi$:*

1. $[\Delta(i) : L_i] = 1$;
2. $P_i \in \mathcal{F}(\Delta)$;
3. $(P_i : \Delta(i)) = 1$ *and* $(P_i : \Delta(j)) \neq 0$ *implies that* $j \sqsupseteq i$.

Proof. We provide a sketch of the proof. Suppose $(B, \Phi, \sqsubseteq)$ satisfies conditions 1 to 3 in the statement of the proposition. In order to prove that $(B, \Phi, \sqsubseteq)$ is quasihereditary we need to show that the poset $(\Phi, \sqsubseteq)$ is adapted to B. So let M be in $\operatorname{mod} B$ and suppose that $\operatorname{Top} M \cong L_i$ and $\operatorname{Soc} M \cong L_j$, with i and j incomparable in $(\Phi, \sqsubseteq)$. The module M is isomorphic to a quotient of P_i. In particular, $\operatorname{Soc} M \cong L_j$ is a composition factor of P_i, but not a composition factor of $\Delta(i)$. Recall that $\Delta(i) = P_i / \operatorname{Tr}\left(\mathcal{P}_{\Phi_{\not\sqsubseteq i}}, P_i\right)$. By the previous observations, the socle of M is a composition factor of $\operatorname{Tr}\left(\mathcal{P}_{\Phi_{\not\sqsubseteq i}}, P_i\right)$. Hence, there must be some simple module in the top of $\operatorname{Tr}\left(\mathcal{P}_{\Phi_{\not\sqsubseteq i}}, P_i\right)$ which appears as a composition factor of M. By condition 3, the module $\operatorname{Tr}\left(\mathcal{P}_{\Phi_{\not\sqsubseteq i}}, P_i\right)$ must be generated by projectives P_k, with $k \sqsupset i$. As a consequence, $[M : L_k] \neq 0$ for some $k \sqsupset i$.

Conversely, suppose that B is quasihereditary with respect to the poset $(\Phi, \sqsubseteq)$. We need to show that B satisfies property 3 in the statement of the proposition. So let i be in Φ and consider the module P_i. We have that $\Delta(i) = P_i / \operatorname{Tr}\left(\mathcal{P}_{\Phi_{\not\sqsubseteq i}}, P_i\right)$. Because $(\Phi, \sqsubseteq)$ is adapted to B then $\operatorname{Tr}\left(\mathcal{P}_{\Phi_{\not\sqsubseteq i}}, P_i\right)$ can be alternatively described as the largest submodule of P_i generated by projectives P_j, with $j \sqsupset i$. One can see this directly by the definition of trace (see Remark 1.4.1) and of adapted poset, or by using Proposition 1.4.6. If i is a maximal element in Φ then $\operatorname{Tr}\left(\mathcal{P}_{\Phi_{\not\sqsubseteq i}}, P_i\right) = 0$ and P_i satisfies property 3. For an arbitrary element i in Φ, the module $\operatorname{Tr}\left(\mathcal{P}_{\Phi_{\not\sqsubseteq i}}, P_i\right)$ is generated by projectives P_j, with $j \sqsupset i$. So the statement follows by descending induction on the poset $(\Phi, \sqsubseteq)$ (using part 5 of Lemma 1.4.8). □

Let $(\Phi, \sqsubseteq)$ be a poset. A subset Φ' of Φ is said to be an *ideal* of $(\Phi, \sqsubseteq)$ if $i \sqsubseteq j$, $j \in \Phi'$, implies that i lies in Φ'. Ideals give rise to new quasihereditary algebras.

Proposition 1.4.13 ([20], [29])**.** *Let $(B, \Phi, \sqsubseteq)$ be a quasihereditary algebra and let Φ' be an ideal of $(\Phi, \sqsubseteq)$. Then:*

1. *the algebra $B/\operatorname{Tr}(\mathcal{P}_{\Phi-\Phi'}, B)$ is a quasihereditary algebra with respect to the poset $(\Phi', \sqsubseteq)$;*
2. *the algebra $\operatorname{End}_B\left(\bigoplus_{i\in\Phi-\Phi'} P_i\right)^{op}$ is quasihereditary with respect to the poset $(\Phi - \Phi', \sqsubseteq)$.*

1.4.3 The tilting modules and the Ringel dual

Let B be a quasihereditary algebra with respect to $(\Phi, \sqsubseteq)$, and let i be in Φ. It was proved by Ringel in [50] (see also Donkin, [22]) that there is a unique indecomposable B-module $T(i)$ (up to isomorphism) which has both a Δ- and a ∇-filtration, with one composition factor labelled by i, and all the other composition factors labelled by j, $j \sqsubset i$.

It is now standard to refer to a module in $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$ as a *tilting module*. Let T be the direct sum of the modules $T(i)$, $i \in \Phi$. This module is called the *characteristic module* in [50], and it is such that $\operatorname{add} T = \mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$. The modules $T(i)$ have remarkable properties.

Theorem 1.4.14 ([50])**.** *Let $(B, \Phi, \sqsubseteq)$ be a quasihereditary algebra. For every $i \in \Phi$ there is a short exact sequence*

$$0 \longrightarrow \Delta(i) \xrightarrow{\phi} T(i) \longrightarrow X(i) \longrightarrow 0 \quad , \tag{1.4}$$

with ϕ a left minimal $\mathcal{F}(\nabla)$-approximation and with $X(i)$ a module lying in

$$\mathcal{F}(\{\Delta(j) : j \sqsubset i\}).$$

Dually, there is an exact sequence

$$0 \longrightarrow Y(i) \longrightarrow T(i) \xrightarrow{\psi} \nabla(i) \longrightarrow 0 \quad , \tag{1.5}$$

with ψ a right minimal $\mathcal{F}(\Delta)$-approximation and with $Y(i)$ a module lying in

$$\mathcal{F}(\{\nabla(j) : j \sqsubset i\}).$$

For $i, j \in \Phi$, $i \neq j$, the indecomposable modules $T(i)$ and $T(j)$ are nonisomorphic, so the endomorphism algebra of T is a basic algebra. Indeed,

$$\mathrm{End}_B(T)^{op}$$

is quasihereditary with respect to the order opposite to the quasihereditary order of B ([50]). This endomorphism algebra, investigated by Ringel in [50], is called the *Ringel dual* of B, and we shall denote it by $\mathcal{R}(B)$. It was shown in [50] that $\mathcal{R}(\mathcal{R}(B)) \cong B$ for B basic.

Denote by P'_i the projective indecomposable $\mathcal{R}(B)$-module $\mathrm{Hom}_B(T, T(i))$ and let L'_i be its top. Let Q'_i represent the injective $\mathcal{R}(B)$-module with socle L'_i. Denote the standard, the costandard and the summands of the characteristic $\mathcal{R}(B)$-module T' accordingly, by $\Delta'(i)$, $\nabla'(i)$ and $T'(i)$, respectively.

Theorem 1.4.15 ([50])**.** *Let* $(B, \Phi, \sqsubseteq)$ *be a quasihereditary algebra. Then* $\mathcal{R}(B)$ *is quasihereditary with respect to* $(\Phi, \sqsubseteq^{op})$*. Moreover, the restriction of the functor*

$$\mathrm{Hom}_B(T, -) : \mathrm{mod}\, B \longrightarrow \mathrm{mod}\, \mathcal{R}(B)$$

to $\mathcal{F}(\nabla)$ *yields an equivalence between the categories* $\mathcal{F}(\nabla)$ *and* $\mathcal{F}(\Delta')$*. Similarly, the functor*

$$D \circ \mathrm{Hom}_B(-, T) : \mathrm{mod}\, B \longrightarrow \mathrm{mod}\, \mathcal{R}(B),$$

where D *is the standard duality, induces an equivalence between the categories* $\mathcal{F}(\Delta)$ *and* $\mathcal{F}(\nabla')$*.*

Remark 1.4.16. Since both $\mathrm{Ext}^1_B(T, \mathcal{F}(\nabla))$ and $\mathrm{Ext}^1_B(\mathcal{F}(\Delta), T)$ vanish (see part 2 of Lemma 1.4.8), one easily sees that: $\mathrm{Hom}_B(T, -)$ maps short exact sequences in $\mathrm{mod}\, B$ with modules in $\mathcal{F}(\nabla)$ to short exact sequences in $\mathrm{mod}\, \mathcal{R}(B)$ with modules in $\mathcal{F}(\Delta')$; $D \circ \mathrm{Hom}_B(-, T)$ maps short exact sequences in $\mathrm{mod}\, B$ with modules in $\mathcal{F}(\Delta)$ to short exact sequences in $\mathrm{mod}\, \mathcal{R}(B)$ with modules in $\mathcal{F}(\nabla')$.

Moreover, the following holds (see the proof of Theorem 6 and Lemma 7 in [50])

$$\begin{array}{ll} \mathrm{Hom}_B(T, T(i)) = P'_i, & D(\mathrm{Hom}_B(T(i), T)) = Q'_i, \\ \mathrm{Hom}_B(T, \nabla(i)) = \Delta'(i), & D(\mathrm{Hom}_B(\Delta(i), T)) = \nabla'(i), \\ \mathrm{Hom}_B(T, Q_i) = T'(i), & D(\mathrm{Hom}_B(P_i, T)) = T'(i). \end{array} \tag{1.6}$$

Chapter 2

Ultra strongly quasihereditary algebras and the ADR algebra

2.1 Overview of the chapter

A prototype for quasihereditary algebras are the Schur algebras, whose highest weight theory is that of general linear groups. They are the endomorphism algebras of certain modules over the group algebra of a symmetric group, and the algebra of the symmetric group can be seen as an idempotent subalgebra of the Schur algebra.

Thus it seems natural that one can study an Artin algebra A by realising it as $(\xi R \xi, \xi)$ with R quasihereditary and ξ an idempotent in R. In [5], Auslander gave an explicit construction of an algebra $\tilde{R}_A$ and an idempotent $\xi \in \tilde{R}_A$ for every Artin algebra A, such that $\tilde{R}_A$ has finite global dimension, and A is isomorphic to $(\xi \tilde{R}_A \xi, \xi)$. In [18], Dlab and Ringel showed that this algebra $\tilde{R}_A$ is in fact quasihereditary. This may be rephrased by saying that any such A has an associated highest weight theory.

In this chapter, we study the basic algebra R_A of $\tilde{R}_A$, where A is an Artin algebra. We propose to call R_A the Auslander–Dlab–Ringel algebra (ADR algebra) of A. We show that R_A satisfies the following two properties:

(A1) $\operatorname{Rad} \Delta(i)$ lies in $\mathcal{F}(\Delta)$;

(A2) if $\operatorname{Rad} \Delta(i) = 0$ then the corresponding indecomposable injective module Q_i has a filtration by standard modules (in other words, Q_i is tilting).

This motivates the following central definition in this dissertation. Let B be a quasihereditary algebra with respect to a poset $(\Phi, \sqsubseteq)$. We say that B is right ultra strongly quasihereditary (RUSQ, for short) if it satisfies (A1) and (A2). This class of algebras is closed under Morita equivalence of quasihereditary algebras, since axioms (A1) and (A2) are expressed in terms of highest weight structures and of internal categorical constructions. By a result of Dlab and Ringel ([19]), condition (A1) holds if and only if the category of modules with a Δ-filtration is closed under submodules, and the algebras with this property were named “right strongly quasihereditary algebras” ([51]).

We prove several results for RUSQ algebras, and for their Ringel duals. In particular, we show that the standard modules are uniserial, and that one can label the simple modules in a natural way by pairs (i, j) so that $\Delta(i, j)$ has radical $\Delta(i, j+1)$ for $1 \leq j < l_i$ and $\Delta(i, l_i)$ is simple. As a main contribution of Section 2.5, we will prove the following (which corresponds to Theorem 2.5.8 and Proposition 2.5.11).

Theorem. *Let B be a RUSQ algebra. The injective hull Q_{i,l_i} of the simple B-module with label (i, l_i) has both a Δ- and a ∇-filtration. Moreover, the chain of inclusions*

$$0 \subset T(i, l_i) \subset \cdots \subset T(i, j) \subset \cdots \subset T(i, 1) = Q_{i,l_i},$$

where $T(i, j)$ is the tilting module corresponding to the label (i, j), is the unique ∇-filtration of Q_{i,l_i}. For $1 \leq j < l_i$, the injective hull $Q_{i,j}$ of the simple module with label (i, j) is isomorphic to $Q_{i,l_i}/T(i, j+1)$.

Most of the content of this chapter was published in [16], the main addition being the reformulation of the definition of RUSQ algebra. This chapter is organised as follows. Section 2.2 contains background on the ADR algebra. In Section 2.3, we study the standard R_A-modules corresponding to the quasihereditary order $(\Lambda, \trianglelefteq)$ of [18]. We prove that the uniserial projective R_A-modules described by Smalø in [57] are indeed standard modules with respect to $(\Lambda, \trianglelefteq)$. In Section 2.4, we show that the algebra R_A is quasihereditary with respect to $(\Lambda, \trianglelefteq)$ – our proof is different from that in [18]. Section 2.5 introduces ultra strongly quasihereditary algebras. We prove the result on the labelling described previously, we construct the injective modules for these algebras and we prove Theorem 2.5.8. In Section 2.6 we show that the Ringel dual $\mathcal{R}(B)^{op}$ of B is a RUSQ algebra whenever B is RUSQ. In Section 2.7 we determine a presentation of R_A by quiver and relations when A is a certain Brauer tree algebra, which occurs for example in the representation theory of the symmetric group.

2.2 The ADR algebra of A

Fix an Artin algebra A. Given a module M in mod A, we shall denote its *Loewy length* by $\mathrm{LL}(M)$, that is, $\mathrm{LL}(M)$ is the minimal natural number such that $\mathrm{Rad}^{\mathrm{LL}(M)} M = 0$.

Let A have Loewy length L (as a left module). We want to study the basic version of the endomorphism algebra of

$$\bigoplus_{j=1}^{L} A/\operatorname{Rad}^j A.$$

This will have multiplicities in general.

Let $\{P_1, \ldots, P_n\}$ be a complete irredundant set of projective indecomposable A-modules and let l_i be the Loewy length of P_i. Define

$$G = G_A := \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{l_i} P_i/\operatorname{Rad}^j P_i. \tag{2.1}$$

The modules $P_i/\operatorname{Rad}^j P_i$ are indecomposable and pairwise nonisomorphic, and these are precisely the indecomposable summands of $\bigoplus_{j=1}^{L} A/\operatorname{Rad}^j A$ (up to isomorphism).

The algebra

$$R = R_A := \operatorname{End}_A(G)^{op},$$

which we call the *Auslander–Dlab–Ringel algebra of* A (ADR algebra of A), is then a basic algebra of

$$\tilde{R}_A := \operatorname{End}_A(\bigoplus_{j=1}^{L} A/\operatorname{Rad}^j A)^{op}.$$

The projective indecomposable R-modules are given by

$$P_{i,j} := \operatorname{Hom}_A\left(G, P_i/\operatorname{Rad}^j P_i\right),$$

for $1 \leq i \leq n$, $1 \leq j \leq l_i$. Let $\xi \in R$ be the idempotent corresponding to the summand $\bigoplus_{i=1}^{n} P_{i,l_i}$ of R. Notice that $\xi R \xi$ is a basic algebra of A.

Denote the simple quotient of $P_{i,j}$ by $L_{i,j}$ and define

$$\Lambda := \{(i,j) : 1 \leq i \leq n, 1 \leq j \leq l_i\}, \tag{2.2}$$

so that Λ labels the simple R-modules.

Recall the properties of the Hom functors described in Subsection 1.2.3. Since G generates A, the functor $\operatorname{Hom}_A(G, -)$ has rather nice properties. In this case, the functor

$$\operatorname{Hom}_A(G, -) : \operatorname{mod} A \longrightarrow \operatorname{mod} R$$

is fully faithful and it is right adjoint to the exact functor $\operatorname{Hom}_R(\operatorname{Hom}_A(G, A), -)$. This implies that $\operatorname{Hom}_A(G, -)$ preserves injectives (see for instance [61, Proposition 2.3.10]). A detailed account of the properties of this adjunction can be found in [5, §8–§10]. According to Proposition 1.2.4, the restriction of $\operatorname{Hom}_A(G, -)$ to $\operatorname{add} G$ yields an equivalence between the categories $\operatorname{add} G$ and $\operatorname{proj} R$.

2.3 The standard modules

Following the notation introduced in Section 2.2, recall that the set

$$\Lambda = \{(i,j) : 1 \leq i \leq n,\, 1 \leq j \leq l_i\}$$

labels the simple modules over the ADR algebra R. Define a partial order, $\trianglelefteq$, on Λ by

$$(i,j) \triangleleft (k,l) \Leftrightarrow j > l. \tag{2.3}$$

It is useful to call to mind the concepts and notation introduced in Subsections 1.4.1 and 1.4.2, namely the definition of a standard module and of a quasihereditary algebra (see (1.2) and Definition 1.4.9).

We shall see, in Section 2.4, that the ADR algebra R is quasihereditary with respect to the poset $(\Lambda, \trianglelefteq)$. In this section, we describe the standard R-modules $\Delta\,(i,j)$ with respect to $(\Lambda, \trianglelefteq)$. For this, two ingredients are needed.

The next result, due to Smalø, is crucial. Recall that a module M is said to be *uniserial* if it has a unique composition series.

Proposition 2.3.1 ([57, Proposition 2.1])**.** *The modules $P_{1,1}, \dots, P_{n,1}$ form a complete irredundant list of projective R-modules without proper projective submodules. Each projective $P_{i,1}$ is uniserial with Loewy length l_i and, for every (i,j) in Λ, we have the following short exact sequences*

$$0 \longrightarrow \operatorname{Hom}_A\left(G, \operatorname{Rad} P_i/\operatorname{Rad}^j P_i\right) \longrightarrow P_{i,j} \longrightarrow \operatorname{Rad}^{j-1} P_{i,1} \longrightarrow 0 \quad .$$

Corollary 2.3.2. *For $1 \leq j \leq l_i$, the module $\operatorname{Rad}^{j-1} P_{i,1}$ is uniserial and has composition factors $L_{i,j}, \cdots, L_{i,l_i}$, labelled from the top to the socle.*

Proof. By Proposition 2.3.1, the projective indecomposable module $P_{i,1}$ has Loewy length l_i and is uniserial. Thus, the module $\operatorname{Rad}^{j-1} P_{i,1}$ is also uniserial and has Loewy length $l_i - j + 1$. Note that $\operatorname{Rad}^k(\operatorname{Rad}^{j-1} P_{i,j}) = \operatorname{Rad}^{k+j-1} P_{i,j}$. By Proposition 2.3.1, this module has a simple top isomorphic to $L_{i,k+j}$, for $0 \leq k \leq l_i - j$. $\square$

The next lemma will also be used to determine the structure of the standard R-modules. Its proof can be found in [5], within the proof of Proposition 10.2. Alternatively, see Lemma 3.3.10, which will be proved in Chapter 3 – this is a stronger version of the next result.

Lemma 2.3.3. *Let M be in* $\operatorname{mod} A$. *There is an epic $\varepsilon : X_0 \longrightarrow M$, with X_0 in* $\operatorname{add} G$ *satisfying* $\mathrm{LL}(X_0) = \mathrm{LL}(M)$, *such that* $\operatorname{Hom}_A(G, \varepsilon)$ *is the projective cover of* $\operatorname{Hom}_A(G, M)$ *in* $\operatorname{mod} R$.

We now have all the necessary results to describe the standard modules over the ADR algebra. For the proof of Proposition 2.3.4, recall the definition of trace given in Example 1.3.2. If B is an algebra endowed with a labelling poset $(\Phi, \sqsubseteq)$ (as in Subsection 1.4.1), then the standard module $\Delta(i)$ is defined as the quotient $P_i / \operatorname{Tr}(\bigoplus_{j:\, j \not\sqsubseteq i} P_j, P_i)$, and this is the largest factor module of P_i whose composition factors are all of the form L_j, with $j \sqsubseteq i$.

Proposition 2.3.4. *For every (i, j) in Λ*

$$\Delta(i, j) \cong \operatorname{Rad}^{j-1} P_{i,1},$$

and there are short exact sequences

$$0 \longrightarrow \operatorname{Hom}_A\big(G, \operatorname{Rad} P_i / \operatorname{Rad}^j P_i\big) \longrightarrow P_{i,j} \longrightarrow \Delta(i, j) \longrightarrow 0 \ . \tag{2.4}$$

The standard R-modules are uniserial. The standard module $\Delta(i, j)$ has Loewy length $l_i - j + 1$, and it has composition factors $L_{i,j}, \ldots, L_{i,l_i}$ (ordered from the top to the socle).

Proof. By Proposition 2.3.1, the module $\operatorname{Rad}^{j-1} P_{i,1}$ is uniserial and it is a quotient of $P_{i,j}$. According to Corollary 2.3.2, $\operatorname{Rad}^{j-1} P_{i,1}$ has composition factors $L_{i,j}, \ldots, L_{i,l_i}$ (ordered from the top to the socle). By the definition of standard module and of the poset $(\Lambda, \trianglelefteq)$, there must be an epic f from $\Delta(i, j)$ to $\operatorname{Rad}^{j-1} P_{i,1}$. Therefore we have the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Tr}\Big(\bigoplus_{(k,l):\,(k,l) \ntrianglelefteq (i,j)} P_{k,l}, P_{i,j}\Big) & \longrightarrow & P_{i,j} & \longrightarrow & \Delta(i, j) & \longrightarrow & 0 \\
 & & \uparrow \exists g & & \| & & \downarrow f & & \\
0 & \longrightarrow & \operatorname{Hom}_A\big(G, \operatorname{Rad} P_i / \operatorname{Rad}^j P_i\big) & \longrightarrow & P_{i,j} & \longrightarrow & \operatorname{Rad}^{j-1} P_{i,1} & \longrightarrow & 0
\end{array}.$$

Furthermore, since $\mathrm{LL}(\operatorname{Rad} P_i / \operatorname{Rad}^j P_i) = j - 1$, it follows from Lemma 2.3.3 that $\operatorname{Hom}_A\big(G, \operatorname{Rad} P_i / \operatorname{Rad}^j P_i\big)$ is generated by projectives $P_{k,l}$, such that $l < j$ (so $(k, l) \ntrianglelefteq (i, j)$). By the definition of trace, the inclusion map is an injection of $\operatorname{Hom}_A\big(G, \operatorname{Rad} P_i / \operatorname{Rad}^j P_i\big)$ into $\operatorname{Tr}(\bigoplus_{(k,l):\,(k,l) \ntrianglelefteq (i,j)} P_{k,l}, P_{i,j})$. Hence the composition of g with this inclusion is one-to-one. But then the monic g must be an isomorphism. Note that $\operatorname{Ker} f \cong \operatorname{Coker} g$, so the epic f must be an isomorphism as well. □

Observe that

$$\operatorname{Rad}\Delta(i,j) = \operatorname{Rad}\left(\operatorname{Rad}^{j-1} P_{i,1}\right) = \begin{cases} \Delta(i,j+1) & \text{if } j < l_i, \\ 0 & \text{if } j = l_i. \end{cases} \tag{2.5}$$

Therefore $\operatorname{Rad}\Delta(i,j)$, which is the unique maximal submodule of $\Delta(i,j)$, belongs to $\mathcal{F}(\Delta)$ for all (i,j) in Λ.

The next lemma can be found in [19, Lemma 2]. We state it for the convenience of the reader.

Lemma 2.3.5. *Let Θ be a set of modules. Assume that for any M in Θ, every maximal submodule of M has a Θ-filtration. Then the category $\mathcal{F}(\Theta)$ is closed under submodules.*

By Lemma 2.3.5 and by the identity (2.5), the subcategory $\mathcal{F}(\Delta)$ of $\operatorname{mod} R$ is closed under submodules. This suggests that there are many R-modules having a Δ-filtration. In fact, the category $\mathcal{F}(\Delta)$ is at least as large as $\operatorname{mod} A$.

Lemma 2.3.6. *The subcategory $\mathcal{F}(\Delta)$ of* $\operatorname{mod} R$ *is closed under submodules. Moreover, for M in* $\operatorname{mod} A$*, the R-module* $\operatorname{Hom}_A(G,M)$ *belongs to $\mathcal{F}(\Delta)$.*

Proof. We prove that $\operatorname{Hom}_A(G,M)$ belongs to $\mathcal{F}(\Delta)$. By Proposition 2.3.4, the result holds if $\operatorname{LL}(M) = 1$, as $\operatorname{Hom}_A(G,L_i) = P_{i,1} = \Delta(i,1)$. Assume that the claim holds for modules with Loewy length $l-1$ and let M have Loewy length l. The functor $\operatorname{Hom}_A(G,-)$ maps the short exact sequence

$$0 \longrightarrow \operatorname{Rad} M \longrightarrow M \longrightarrow M/\operatorname{Rad} M \longrightarrow 0$$

to

$$\begin{array}{ccccc} 0 \to \operatorname{Hom}_A(G,\operatorname{Rad} M) & \to & \operatorname{Hom}_A(G,M) & \longrightarrow & \operatorname{Hom}_A(G,M/\operatorname{Rad} M) \\ & & & \searrow\!\!\!\!\to & \uparrow \\ & & & & \operatorname{Hom}_A(G,M)/\operatorname{Hom}_A(G,\operatorname{Rad} M) \end{array} .$$

By induction, $\operatorname{Hom}_A(G,\operatorname{Rad} M)$ lies in $\mathcal{F}(\Delta)$, and by the initial case, the module $\operatorname{Hom}_A(G,M/\operatorname{Rad} M)$ belongs to $\mathcal{F}(\Delta)$ as well. Since $\mathcal{F}(\Delta)$ is closed under submodules, then

$$\operatorname{Hom}_A(G,M)/\operatorname{Hom}_A(G,\operatorname{Rad} M) \in \mathcal{F}(\Delta).$$

The result follows from the fact that $\mathcal{F}(\Delta)$ is closed under extensions (Lemma 1.4.8). $\square$

2.4 The ADR algebra is quasihereditary

The ADR algebra is quasihereditary with respect to the heredity chain constructed by Dlab and Ringel in [18]. The underlying order in [18] can be shown to be the same as our partial order $(\Lambda, \trianglelefteq)$. Instead of going into details about heredity chains, we give a different proof that R is quasihereditary with respect to $(\Lambda, \trianglelefteq)$. For this, recall the definition of an adapted order – Definition 1.4.2.

Lemma 2.4.1. *The partial order* $(\Lambda, \trianglelefteq)$ *for the simple* R*-modules is an adapted order for* R*.*

Proof. Let N be an indecomposable R-module. Suppose that $\operatorname{Top} N = L_{i,j}$ and $\operatorname{Soc} N = L_{k,l}$, with (i,j) and (k,l) incomparable with respect to $\trianglelefteq$. Then $j = l$ and $i \neq k$, by (2.3). There is a nonzero morphism f and a commutative diagram

$$\begin{array}{ccc} & & P_{k,l} \\ & \swarrow{\scriptstyle \exists\, t_*} & \Big\downarrow{\scriptstyle f} \\ P_{i,l} & \twoheadrightarrow & N \end{array} .$$

Note that $t_* = \operatorname{Hom}_A(G, t)$ for some morphism $t : P_k / \operatorname{Rad}^l P_k \longrightarrow P_i / \operatorname{Rad}^l P_i$ (recall that $\operatorname{Hom}_A(G, -)$ is a full functor, or use Proposition 1.2.4). The map t must be a nonisomorphism since $k \neq i$. So $\operatorname{Im} t$ is generated by a module in

$$\mathcal{C} = \operatorname{add}\left(\bigoplus_{(x,y)\colon y \leq l-1} P_x / \operatorname{Rad}^y P_x \right).$$

By the projectivity of $P_k / \operatorname{Rad}^l P_k$ in $\operatorname{mod}(A / \operatorname{Rad}^l A)$, we conclude that t factors through a module in $\mathcal{C}$. Hence t_* factors through a module in

$$\operatorname{add}\left(\bigoplus_{(x,y)\colon y \leq l-1} P_{x,y} \right).$$

But then N must have a composition factor of the form $L_{x,y}$ for some x and some $y < l$, i.e. for some pair (x,y) such that $(x,y) \triangleright (k,l)$. □

We are now able to prove that the ADR algebra R is quasihereditary with respect to the poset $(\Lambda, \trianglelefteq)$.

Theorem 2.4.2. *The algebra* R *is quasihereditary with respect to* $(\Lambda, \trianglelefteq)$*.*

Proof. We check that $(R, \Lambda, \trianglelefteq)$ satisfies conditions 1 to 3 in Definition 1.4.9. By Lemma 2.4.1, the poset $(\Lambda, \trianglelefteq)$ is adapted to R. Proposition 2.3.4 implies that $[\Delta(i,j) : L_{i,j}] = 1$. Finally, recall that $P_{i,j} = \mathrm{Hom}_A\left(G, P_i / \mathrm{Rad}^j P_i\right)$. By Lemma 2.3.6, the projective indecomposable R-modules lie in $\mathcal{F}(\Delta)$. □

The next theorem, due to Dlab and Ringel ([19], [21, Lemma 4.1*]), is stated for ease of reference. Theorem 2.4.3 is a very useful and informative result.

Theorem 2.4.3. *Let $(B, \Phi, \sqsubseteq)$ be a quasihereditary algebra. The following assertions are equivalent:*

1. $\mathrm{Rad}\,\Delta(i) \in \mathcal{F}(\Delta)$ *for all* $i \in \Phi$*;*
2. $\mathcal{F}(\Delta)$ *is closed under submodules;*
3. *for all i in Φ the module $\nabla(i)$ has injective dimension at most one;*
4. *every module in $\mathcal{F}(\nabla)$ has injective dimension at most one;*
5. *every torsionless module (i.e. every module cogenerated by projectives) belongs to $\mathcal{F}(\Delta)$.*

Following Ringel ([51]), a quasihereditary algebra $(B, \Phi, \sqsubseteq)$ is said to be *right strongly quasihereditary* if one of the equivalent statements in Theorem 2.4.3 holds for B. Dually, B is called *left strongly quasihereditary* if $(B^{op}, \Phi, \sqsubseteq)$ is a right strongly quasihereditary algebra. The general term *strongly quasihereditary algebra* will be used for both left and right strongly quasihereditary algebras.

According to Lemma 2.3.6, the ADR algebra R is right strongly quasihereditary. Compare this statement with Observation (2) in [51]. In this article, Ringel observes that every Artin algebra A can be embedded as an idempotent subalgebra in a left strongly quasihereditary algebra Γ. The algebra Γ in question is obtained by applying Iyama's construction to the regular module A (see [51]).

From now onwards denote the simple quotient of the A-module P_i by L_i and let Q_i be the injective A-module with socle L_i. Similarly, let $Q_{i,j}$ be the injective R-module with socle $L_{i,j}$. We claim that the R-modules Q_{i,l_i} have a Δ-filtration.

Lemma 2.4.4. *The functor* $\mathrm{Hom}_A(G, -)$ *preserves indecomposable modules. In particular,* $Q_{i,l_i} = \mathrm{Hom}_A(G, Q_i)$*, and* $\mathrm{Hom}_A(G, -)$ *preserves injective hulls.*

Proof. The first assertion follows from the fact that $\mathrm{Hom}_A(G,-)$ is a fully faithful functor. As observed previously, the functor $\mathrm{Hom}_A(G,-)$ also preserves injective modules. Note that the inclusion of L_i in Q_i induces a monic from $P_{i,1}$ (whose socle is L_{i,l_i} by Corollary 2.3.2) to $\mathrm{Hom}_A(G,Q_i)$. So indeed $Q_{i,l_i} = \mathrm{Hom}_A(G,Q_i)$.

Let now M be in $\mathrm{mod}\, A$ and suppose that $\mathrm{Soc}\, M = \bigoplus_{j\in J} L_{x_j}$. Then $\bigoplus_{j\in J} P_{x_j,1}$ (whose socle is $\bigoplus_{j\in J} L_{x_j,l_{x_j}}$) is contained in $\mathrm{Hom}_A(G,M)$. Moreover, the functor $\mathrm{Hom}_A(G,-)$ maps the injective hull of the module M to a monic from $\mathrm{Hom}_A(G,M)$ to $\bigoplus_{j\in J} Q_{x_j,l_{x_j}}$, so the statement follows. □

2.5 Ultra strongly quasihereditary algebras

Let B be a quasihereditary algebra with respect to $(\Phi,\sqsubseteq)$. Recall the definition of tilting module given in Subsection 1.4.3. For every $i\in\Phi$ there is a unique indecomposable B-module $T(i)$ (up to isomorphism) which is both Δ- and ∇-good, and has one composition factor labelled by i, and all the other composition factors labelled by j, $j\sqsubset i$. Denote the charateristic B-module by T (so T is the direct sum of the modules $T(i)$), and remember that $\mathrm{add}\, T = \mathcal{F}(\Delta)\cap\mathcal{F}(\nabla)$.

Lemmas 2.3.6 and 2.4.4 imply that the R-modules Q_{i,l_i} belong to $\mathcal{F}(\Delta)\cap\mathcal{F}(\nabla) = \mathrm{add}\, T$. Consequently, every module Q_{i,l_i} is a tilting module.

In this section we:

(I) introduce the class of ultra strongly quasihereditary algebras, which contains the ADR algebras;

(II) for B a right ultra strongly quasihereditary algebra, investigate the injective and the tilting modules – our main results are Proposition 2.5.6, Theorem 2.5.8 and Proposition 2.5.11.

2.5.1 Definition and first properties

Let $(B,\Phi,\sqsubseteq)$ be an arbitrary quasihereditary algebra, as before. Additionally, suppose that B satisfies the following two conditions:

(A1) $\mathrm{Rad}\,\Delta(i)\in\mathcal{F}(\Delta)$ for all $i\in\Phi$ (i.e. B is right strongly quasihereditary);

(A2) $Q_i\in\mathcal{F}(\Delta)$ for all $i\in\Phi$ such that $\mathrm{Rad}\,\Delta(i)=0$.

We call these algebras *right ultra strongly quasihereditary* algebras (RUSQ algebras, for short). We say that $(B,\Phi,\sqsubseteq)$ is a *left ultra strongly quasihereditary algebra* (LUSQ algebra) if $(B^{op}, \Phi, \sqsubseteq)$ is RUSQ. The designation *ultra strongly quasihereditary* will be used generically for RUSQ and LUSQ algebras.

Note that the conditions in Theorem 2.4.3 hold for every RUSQ algebra. Observe that the algebra R_A is RUSQ for every choice of A: R_A satisfies the identity (2.5) and the modules Q_{i,l_i} are Δ-good. However, notice that there are RUSQ algebras which are not isomorphic to R_A for any A.

Example 2.5.1. Consider the path algebra $B = KQ$, where Q is the quiver

$$\overset{n}{\circ} \longrightarrow \overset{n-1}{\circ} \longrightarrow \cdots \longrightarrow \overset{1}{\circ} \quad .$$

The algebra B is quasihereditary with respect to the natural ordering. Besides, B satisfies (A1) and (A2). However it is not difficult to see that B is isomorphic to the quasihereditary algebra R_A for some A if and only if $n = 1$.

Many more examples of ultra strongly quasihereditary algebras will be investigated in Chapter 5. Notice, for instance, that if $(B, \Phi, \sqsubseteq)$ is a RUSQ algebra and if Φ' is an ideal of $(\Phi, \sqsubseteq)$, then the quasihereditary algebra $(B/\operatorname{Tr}(\mathcal{P}_{\Phi-\Phi'}, B), \Phi', \sqsubseteq)$ (as in Proposition 1.4.13) is still RUSQ. Roughly speaking, this holds because the standard and the tilting modules over $B/\operatorname{Tr}(\mathcal{P}_{\Phi-\Phi'}, B)$ coincide with those over B which are labelled by Φ' (see [29, §1.1]).

We now state the fundamental properties of the standard modules over a RUSQ algebra.

Lemma 2.5.2. *Let $(B, \Phi, \sqsubseteq)$ be a quasihereditary algebra satisfying axiom (A1). Then the category $\mathcal{F}(\Delta)$ is closed under submodules, and any nonzero map*

$$f : \Delta(i) \longrightarrow M,$$

with M in $\mathcal{F}(\Delta)$, must be monic.

Proof. Axiom (A1) holds for B if and only if the category $\mathcal{F}(\Delta)$ is closed under submodules. Let B be a quasihereditary algebra satisfying axiom (A1), and consider a nonzero map $f : \Delta(i) \longrightarrow M$, with M in $\mathcal{F}(\Delta)$. This gives rise to a short exact sequence

$$0 \longrightarrow \operatorname{Ker} f \longrightarrow \Delta(i) \longrightarrow \operatorname{Im} f \longrightarrow 0 \quad .$$

Note that both $\operatorname{Ker} f$ and $\operatorname{Im} f$ lie in $\mathcal{F}(\Delta)$, as this category is closed under submodules. According to part 3 of Lemma 1.4.8, this short exact sequence splits. Since $f \neq 0$, then f must be a monomorphism. □

Let $(B, \Phi, \sqsubseteq)$ be a quasihereditary algebra. If $i \in \Phi$ is a maximal element with respect to $\sqsubseteq$, then $\Delta(i)$ is the only standard B-module which has L_i as a composition factor. The converse of this assertion is not true in general.

Lemma 2.5.3. *Let $(B, \Phi, \sqsubseteq)$ be a RUSQ algebra. Let $i_1 \in \Phi$ be such that $\Delta(i_1)$ is the only standard module having L_{i_1} as a composition factor. Then $\Delta(i_1)$ has simple socle, denote it by L_{i_*}. Moreover, $Q_{i_*} = T(i_1)$.*

Proof. Let L_{i_*} be a summand of the socle of $\Delta(i_1)$. By Lemma 2.5.2, $\mathcal{F}(\Delta)$ is closed under submodules. Hence the module L_{i_*} is in $\mathcal{F}(\Delta)$, and we must have $\Delta(i_*) = L_{i_*}$. By axiom (A2), the module Q_{i_*} is in $\mathcal{F}(\Delta)$. There is a nonzero map from $\Delta(i_1)$ to Q_{i_*} and, by Lemma 2.5.2, this map must be monic. That is, the module Q_{i_*} has a submodule M which is isomorphic to $\Delta(i_1)$. In particular, $\operatorname{Soc}\Delta(i_1) = L_{i_*}$. As Q_{i_*} is an indecomposable module in $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$, then $Q_{i_*} = T(k)$ for some k in Φ. We must have $i_1 \sqsubseteq k$ as all composition factors of $T(k)$ are of the form L_l, with $l \sqsubseteq k$.

We claim that $k = i_1$. By Theorem 1.4.14, $\Delta(k)$ may be regarded as a submodule of $T(k)$. Consider the following submodule of $T(k)$:

$$\Delta(k) + M \cong (\Delta(k) \oplus M) / (\Delta(k) \cap M).$$

Suppose, by contradiction, that $i_1 \sqsubset k$. As a consequence, we have that $\Delta(k) \cap M \subseteq \operatorname{Rad}\Delta(k)$. Since $\Delta(i_1)$ is the only standard module having L_{i_1} as a composition factor, then $\Delta(k) \cap M \subseteq \operatorname{Rad} M$. But then $\Delta(k) + M$ is a submodule of $T(k)$ with top $L_k \oplus L_{i_1}$. As $\mathcal{F}(\Delta)$ is closed under submodules, the module $\Delta(k) + M$ belongs to $\mathcal{F}(\Delta)$ and the standard modules $\Delta(k)$ and $\Delta(i_*)$ must appear in its Δ-filtration. Therefore we must have $\Delta(k) \cap M = 0$, so $\Delta(k) \oplus M$ is a submodule of $T(k)$. This cannot happen as $T(k)$ has simple socle L_{i_*} (recall that $T(k) = Q_{i_*}$). Thus $i_1 = k$, as claimed.

We have proved that if $i_1 \in \Phi$ is such that $\Delta(i_1)$ is the only standard module having L_{i_1} as a composition factor, then $\Delta(i_1)$ has simple socle, say L_{i_*}, and $Q_{i_*} = T(i_1)$. □

The following results show that the axioms (A1) and (A2) encapsulate some of the key properties of the ADR algebras. In other words, this abstraction is well suited to the class of ADR algebras.

Lemma 2.5.4. *Let $(B, \Phi, \sqsubseteq)$ be a RUSQ algebra. Let $i_1 \in \Phi$ be such that $\Delta(i_1)$ is the only standard module having L_{i_1} as a composition factor. Then $\Delta(i_1)$ is a uniserial module and every nonzero submodule of $\Delta(i_1)$ is a standard module.*

Proof. Suppose that $\mathrm{LL}(\Delta(i_1)) = l$. We claim that

$$\mathrm{Soc}_t\, \Delta(i_1) \in \Delta,$$

for all $1 \leq t \leq l$. This will imply the statement of the lemma.

We prove this claim by induction on t – note that the claim holds for $t = 1$. In fact, using the notation of Lemma 2.5.3, we have that $\mathrm{Soc}\, \Delta(i_1) = L_{i_*}$, and $L_{i_*} = \Delta(i_*)$ because $\mathcal{F}(\Delta)$ is closed under submodules (see Lemma 2.5.2). Suppose now that $l \geq 2$, let t be such that $1 \leq t \leq l-1$, and assume that $\mathrm{Soc}_t\, \Delta(i_1) = \Delta(r)$ for some r in Φ. We want to prove that $\mathrm{Soc}_{t+1}\, \Delta(i_1)$ lies in Δ. Let L_s be a summand of $\mathrm{Soc}_{t+1}\, \Delta(i_1) / \mathrm{Soc}_t\, \Delta(i_1)$, and let N be the image of the map $P_s \longrightarrow \mathrm{Soc}_{t+1}\, \Delta(i_1)$ which maps the top of P_s to the summand L_s of $\mathrm{Soc}_{t+1}\, \Delta(i_1) / \mathrm{Soc}_t\, \Delta(i_1)$. Notice that:

($\diamondsuit$) N is in $\mathcal{F}(\Delta)$ as this category is closed under submodules; because N has top L_s, then $\Delta(s)$ appears in (the top part of) a Δ-filtration of N; moreover, note that $\mathrm{Rad}\, N = \Delta(r)$.

One of the following situations holds:

(I) $r \sqsubseteq s$;

(II) $r \sqsupset s$;

(III) r and s are not comparable.

We now analyse each of these situations.

Situation (III) cannot happen since there is a B-module with Loewy length 2, with top L_s and socle L_r and, at the same time, $(\Phi, \sqsubseteq)$ is adapted to B (recall Definition 1.4.2).

By the observation ($\diamondsuit$), if condition (II) holds then $\Delta(s) = L_s$. This is because the simple top of the module N has to be isomorphic to $\Delta(s)$.

Suppose now that condition (I) holds. By observation ($\diamondsuit$), we conclude that N must be isomorphic to $\Delta(s)$ in this case. If $\mathrm{Soc}_{t+1}\, \Delta(i_1)$ has simple top then $\mathrm{Soc}_{t+1}\, \Delta(i_1) = N$, so $\mathrm{Soc}_{t+1}\, \Delta(i_1) \in \Delta$ and we are done. Otherwise, the module $\mathrm{Soc}_{t+1}\, \Delta(i_1) / \mathrm{Soc}_t\, \Delta(i_1)$ has some other summand $L_{s'}$, and we may consider the map

$$P_s \oplus P_{s'} \longrightarrow \mathrm{Soc}_{t+1}\, \Delta(i_1)$$

which sends P_s to N and $\mathrm{Top}\, P_{s'}$ to the summand $L_{s'}$ of $\mathrm{Soc}_{t+1}\, \Delta(i_1) / \mathrm{Soc}_t\, \Delta(i_1)$. Let N' be the image of the map above. Note that N' belongs to $\mathcal{F}(\Delta)$ as this class is closed under submodules. Notice that the modules $\Delta(s)$ and $\Delta(s')$ certainly appear in a Δ-filtration of N'. Since $\Delta(s) \cong N \subseteq N'$, and $N'/N \cong L_{s'}$ then we must have $\Delta(s') = L_{s'}$.

We have concluded that, if condition (II) holds, or if condition (I) holds and $\operatorname{Soc}_{t+1} \Delta(i_*) \neq N$, then there is a summand of $\operatorname{Soc}_{t+1} \Delta(i_1) / \operatorname{Soc}_t \Delta(i_1)$ which is isomorphic to a standard module. That is, there is a summand L_u on the $(t+1)^{\text{th}}$ socle layer of $\Delta(i_1)$ satisfying $L_u = \Delta(u)$. By axiom (A2), the module Q_u must be in $\mathcal{F}(\Delta)$. So there is a nonzero morphism

$$\Delta(i_*) \xrightarrow{\pi} \!\!\!\!\rightarrow \Delta(i_*) / \operatorname{Soc}_t \Delta(i_*) \longrightarrow Q_u \quad ,$$

and by Lemma 2.5.2 this map must be monic. This cannot happen because π is a proper epic.

The arguments in the previous paragraph imply that only situation (I) can happen and that, additionally, we must have $\operatorname{Soc}_{t+1} \Delta(i_1) = N = \Delta(s)$. This proves that, for every element i_1 in Φ such that $\Delta(i_1)$ is the only standard module having L_{i_1} as a composition factor, we have that the standard module $\Delta(i_1)$ is uniserial and all of its submodules are standard modules. □

Proposition 2.5.5. *Let $(B, \Phi, \sqsubseteq)$ be a RUSQ algebra. Let Φ_1 be the set of all $i_1 \in \Phi$ such that $\Delta(i_1)$ is the only standard module having L_{i_1} as a composition factor. The following holds:*

1. *every standard module is contained in some standard module $\Delta(i_1)$, $i_1 \in \Phi_1$;*
2. *every nonzero submodule of a standard module is still a standard module, and every standard module is uniserial;*
3. *if i_1 and k_1 are two elements in Φ_1, and $\Delta(i_1)$ and $\Delta(k_1)$ have some composition factor in common, then $i_1 = k_1$.*

Proof. Let $\Delta(i^{(1)})$ be an arbitrary standard module. We want to prove that $\Delta(i^{(1)})$ is uniserial and that all of its submodules are standard modules. If $i^{(1)}$ is such that $\Delta(i^{(1)})$ is the only standard module having $L_{i^{(1)}}$ as composition factor we are done (see Lemma 2.5.4). Otherwise, there is a standard module $\Delta(i^{(2)})$, with $i^{(2)} \neq i^{(1)}$, such that the factor $L_{i^{(1)}}$ appears in its composition series. Note that $i^{(1)} \sqsubset i^{(2)}$. In general, there is a sequence of standard modules $\Delta(i^{(1)}), \ldots, \Delta(i^{(k)}), \ldots$, such that $\Delta(i^{(k)})$ has $L_{i^{(k-1)}}$ as a composition factor and $i^{(k)} \neq i^{(k-1)}$. Such sequence must be finite because the indexes are increasing strictly, that is, $i^{(k-1)} \sqsubset i^{(k)}$. So there is a sequence of standard modules $\Delta(i^{(1)}), \ldots, \Delta(i^{(m)})$, such that $\Delta(i^{(k)})$ has $L_{i^{(k-1)}}$ as a composition factor, $i^{(k)} \neq i^{(k-1)}$, for all $1 \leq k \leq m$, and such that $\Delta(i^{(m)})$ is the only standard B-module having $L_{i^{(m)}}$ as a composition factor (i.e. $i^{(m)} \in \Phi_1$). By Lemma 2.5.4, the module $\Delta(i^{(m)})$ is uniserial and all its submodules are standard modules. Since $L_{i_{m-1}}$ is a composition factor of $\Delta(i^{(m)})$, then we may embedded the standard module $\Delta(i^{(m-1)})$ in $\Delta(i^{(m)})$. By applying this reasoning inductively, we conclude that $\Delta(i^{(1)})$ is a submodule of the uniserial module $\Delta(i^{(m)})$. This proves part 1 and part 2 in the statement of the proposition (using Lemma 2.5.4).

In order to prove part 3, let i_1 and k_1 be in Φ_1, and suppose that $\Delta(i_1)$ and $\Delta(k_1)$ have some composition factor in common, say L_j. By part 2, $\Delta(j)$ can be embedded in both $\Delta(i_1)$ and $\Delta(k_1)$, and moreover these two module have the same simple socle, say L_{i_*}. Lemma 2.5.3 implies that $Q_{i_*} \cong T(i_1) \cong T(k_1)$. Thus $i_1 = k_1$. □

Let $(B, \Phi, \sqsubseteq)$ be a RUSQ algebra. Suppose i_1 belongs to the set Φ_1 (as in Proposition 2.5.5). The module $\Delta(i_1)$ is uniserial. Assume $\Delta(i_1)$ has Loewy length l_i and, by analogy with the ADR algebra R, let $L_{i_1}, \ldots, L_{i_{l_i}}$ be the composition factors of $\Delta(i_1)$, ordered from the top to the socle. We may relabel the simple B-modules as (i, j), where, for every i_1 in Φ_1, the label i_1 is replaced by $(i, 1)$, and the remaining labels i_j (as before) are replaced by (i, j). By part 1 of Proposition 2.5.5, every simple B-module has been given such a label. Furthermore, part 3 of Proposition 2.5.5 assures that this relabelling is well defined. Note that this relabelling is consistent with the labels chosen for the simple modules over the ADR algebra. From now onwards we will use this new labelling for the simple B-modules. I.e., we shall assume (unless otherwise stated) that $(B, \Phi, \sqsubseteq)$ denotes a RUSQ algebra and that

$$\Phi = \{(i, j) : 1 \leq i \leq n, 1 \leq j \leq l_i\}. \tag{2.6}$$

So $L_{i,j}$, $P_{i,j}$, $Q_{i,j}$, $\Delta(i, j)$, $\nabla(i, j)$, $T(i, j)$ and T will be the naturally expected B-modules. The following proposition summarises our conclusions about RUSQ algebras.

Proposition 2.5.6. *Let $(B, \Phi, \sqsubseteq)$ be a RUSQ algebra. Using the labelling introduced in* (2.6)*, the following holds:*

1. *$\mathcal{F}(\Delta)$ is closed under submodules and the standard modules are uniserial;*

2. *$\operatorname{Rad} \Delta(i, j) = \Delta(i, j+1)$ for $j < l_i$, and $\Delta(i, l_i) = L_{i,l_i}$;*

3. *$Q_{i,l_i} = T(i, 1)$;*

4. *for $M \in \mathcal{F}(\Delta)$, the number of standard modules appearing in a Δ-filtration of M is given by $\sum_{i=1}^{n}[M : L_{i,l_i}]$*

5. *a module M belongs to $\mathcal{F}(\Delta)$ if and only if $\operatorname{Soc} M$ is a (finite) direct sum of modules of type L_{i,l_i};*

Proof. Part 1 follows from Lemma 2.5.2 and Proposition 2.5.5. Parts 2 and 3 are just a reformulation of our the previous conclusions in terms of the relabelling introduced in (2.6). Note that part 4 is a consequence of part 2.

We now prove part 5. If M is in $\mathcal{F}(\Delta)$ then, by part 1, $\operatorname{Soc} M$ lies in $\mathcal{F}(\Delta)$ and part 2 implies that $\operatorname{Soc} M$ is a direct sum of modules of type L_{i,l_i}. Conversely, if the socle of a module M is a direct sum of simples L_{i,l_i}, then the injective hull of M is a direct sum of injective modules of type Q_{i,l_i}. Therefore M lies in $\mathcal{F}(\Delta)$ by part 3 and part 1. □

2.5.2 The structure of an ultra strongly quasihereditary algebra

Let $(B, \Phi, \sqsubseteq)$ be a RUSQ algebra, and consider an injective B-module of type Q_{i,l_i}. By Proposition 2.5.6, Q_{i,l_i} is isomorphic to $T(i, 1)$. As we shall see shortly, every module $T(i, j)$ may be determined recursively from $T(i, 1)$. The next lemma will be useful when proving this claim.

Lemma 2.5.7. *Let $(B, \Phi, \sqsubseteq)$ be an arbitrary quasihereditary algebra. For $i \in \Phi$ consider the short exact sequence*

$$0 \longrightarrow Y(i) \longrightarrow T(i) \xrightarrow{\psi} \nabla(i) \longrightarrow 0 \quad , \tag{2.7}$$

as in (1.5), Theorem 1.4.14 (i.e. with ψ a right minimal $\mathcal{F}(\Delta)$-approximation of $\nabla(i)$ and with $Y(i)$ a module lying in $\mathcal{F}(\{\nabla(j) : j \sqsubset i\})$). Then:

1. *$\operatorname{Rad}\Delta(i)$ is a submodule of $Y(i)$;*

2. *for every morphism $f : T(i) \longrightarrow \nabla(i)$, there is a map h in the division ring $\operatorname{End}_B(\nabla(i))$ such that $f = h \circ \psi$;*

3. *if $M \subseteq T(i)$, with M a module in $\mathcal{F}(\nabla)$ and $T(i)/M$ a costandard module, then $T(i)/M = \nabla(i)$ and $M = Y(i)$.*

Proof. Consider the exact sequence

$$0 \longrightarrow \Delta(i) \xrightarrow{\phi} T(i) \longrightarrow X(i) \longrightarrow 0 \ , \tag{2.8}$$

as in (1.4), Theorem 1.4.14, where $X(i)$ lies $\mathcal{F}(\{\Delta(j) : j \sqsubset i\})$. We may regard $\Delta(i)$ as a submodule of $T(i)$. The image of $\Delta(i)$ under ψ must be the socle of $\nabla(i)$, since L_i occurs only once as a composition factor of $T(i)$. This proves part 1.

Now apply the functor $\mathrm{Hom}_B(-, \nabla(i))$ to (2.8). We have $\mathrm{Hom}_B(X(i), \nabla(i)) = 0$, as L_i is not a composition factor of $X(i)$. Because of this, and also because $\mathrm{Ext}^1_B(\mathcal{F}(\Delta), \mathcal{F}(\nabla)) = 0$ (see Lemma 1.4.8), we get an isomorphism

$$\mathrm{Hom}_B(T(i), \nabla(i)) \longrightarrow \mathrm{Hom}_B(\Delta(i), \nabla(i))$$

of S-modules, where $S := \mathrm{End}_B(\nabla(i))$ is a division ring. By Lemma 1.4.11, the module $\mathrm{Hom}_B(\Delta(i), \nabla(i))$ is 1-dimensional over S. So $\mathrm{Hom}_B(T(i), \nabla(i))$ is 1-dimensional over S as well, which proves part 2.

For part 3, note that the epic $f : T(i) \longrightarrow T(i)/M$ must be a right $\mathcal{F}(\Delta)$-approximation of $T(i)/M$, as $\mathrm{Ext}^1_B(\mathcal{F}(\Delta), M) = 0$ (consult Subsection 1.2.2 for the definition of right approximation). Since $T(i)$ is an indecomposable module, the map f is in fact a right minimal $\mathcal{F}(\Delta)$-approximation of $T(i)/M$. Suppose that $T(i)/M = \nabla(j)$. So both f and $\psi : T(j) \longrightarrow \nabla(j)$ are right minimal $\mathcal{F}(\Delta)$-approximations of $\nabla(j)$. As a consequence, $T(j)$ and $T(i)$ must be isomorphic (see Subsection 1.2.2), so $j = i$. If we look at $Y(i)$ as a submodule of $T(i)$, then part 2 implies that $\iota = \iota' \circ t$, where t is an isomorphism and $\iota : Y(i) \longrightarrow T(i)$, $\iota' : M \longrightarrow T(i)$ are the inclusion maps. Thus $M = Y(i)$. □

We are now in position of proving one of the main results of this chapter.

Theorem 2.5.8. *Let $(B, \Phi, \sqsubseteq)$ be a RUSQ algebra. Then $Q_{i,l_i} = T(i,1)$ and, for every $(i,j) \in \Phi$, we have the following short exact sequence*

$$0 \longrightarrow T(i,j+1) \longrightarrow T(i,j) \xrightarrow{\psi} \nabla(i,j) \longrightarrow 0 \ , \tag{2.9}$$

where $T(i, l_i + 1) := 0$. In particular,

$$0 \subset T(i,l_i) \subset \cdots \subset T(i,j) \subset \cdots \subset T(i,1) = Q_{i,l_i} \tag{2.10}$$

is the unique ∇-filtration of $T(i,1)$.

Proof. By part 3 of Proposition 2.5.6, we must have $Q_{i,l_i} = T(i,1)$. We will prove by induction on k, that there is a filtration

$$T(i,k) \subset T(i,k-1) \subset \cdots \subset T(i,1) = Q_{i,l_i}.$$

For $k = 1$ the claim is obvious. Suppose that the claim holds for all $k \le j$. We get that $T(i,j) \subseteq T(i,1) = Q_{i,l_i}$. Consider the short exact sequence

$$0 \longrightarrow Y(i,j) \longrightarrow T(i,j) \xrightarrow{\psi} \nabla(i,j) \longrightarrow 0$$

(as in (1.5), with $Y(i,j) \in \mathcal{F}(\{\nabla(k,l) : (k,l) \sqsubset (i,j)\})$). Suppose that $j \neq l_i$. The map ψ cannot be an isomorphism, as $\operatorname{Soc} \nabla(i,j) = L_{i,j}$ and $\operatorname{Soc} T(i,j) = L_{i,l_i}$. Since $Y(i,j) \subseteq T(i,j)$, we get that $\operatorname{Soc} Y(i,j) = L_{i,l_i}$. Therefore $Y(i,j)$ is indecomposable. Since $\mathcal{F}(\Delta)$ is closed under submodules, we must have $Y(i,j) \in \mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$. Thus $Y(i,j) = T(i,l)$, for some $1 \le l \le l_i$ (note that $\Delta(k,l) \subseteq T(k,l)$, so $T(k,l)$ must have the summand L_{k,l_k} in its socle). From Lemma 2.5.7 and Proposition 2.5.6, we also know that $\operatorname{Rad} \Delta(i,j) = \Delta(i,j+1)$ is contained in $Y(i,j) = T(i,l)$. Hence $(i,j+1) \sqsubseteq (i,l)$, so $j+1 \ge l$. We cannot have $l \le j$, otherwise, as $\Delta(i,l)$ is a submodule of $T(i,l)$, $L_{i,j}$ would be a composition factor of $Y(i,j)$. Thus $l = j+1$ and $Y(i,j) = T(i,j+1)$. This proves the existence of the chain of modules (2.10), and shows that $T(i,j)/T(i,j+1) \cong \nabla(i,j)$ for $1 \le j < l_i$.

In order to prove that $T(i,l_i) \cong \nabla(i,l_i)$, observe that

$$Y(i,l_i) \subseteq T(i,l_i) \subseteq Q_{i,l_i}.$$

But then $Y(i,l_i) = 0$, otherwise $Y(i,l_i)$ would have socle L_{i,l_i}. Therefore $T(i,l_i) \cong \nabla(i,l_i)$, and we get a ∇-filtration as in (2.10), with factors as described in (2.9). Part 3 of Lemma 2.5.7 assures the uniqueness of this ∇-filtration. □

Remark 2.5.9. Let $1 \le j < j' \le l_i$. Then $T(i,j')$ is a submodule of $T(i,j)$. We claim that the B-module $T(i,j)/T(i,j')$ is indecomposable. In order to see this, note that $T(i,j)/T(i,j')$ belongs to $\mathcal{F}(\nabla)$. To be precise, the module $T(i,j)/T(i,j')$ must have a unique ∇-filtration as this is the case of $T(i,1)$ (look at (2.10)). Since $\mathcal{F}(\nabla)$ is closed under direct summands, every module having a unique ∇-filtration must be indecomposable.

From the filtration (2.10), it is not difficult to conclude that $T(i,j)$ is isomorphic to $\operatorname{Rej}(Q_{i,l_i}, \bigoplus_{(k,l):\,(k,l) \sqsupset (i,j)} Q_{k,l})$ (using the property of the ∇-filtrations stated in Proposition 1.4.6). This alternative characterisation of the tilting modules $T(i,j)$ will be useful in Chapter 3.

Lemma 2.5.10. *Let $(B, \Phi, \sqsubseteq)$ be a RUSQ algebra. Then*

$$T(i,j) = \operatorname{Rej}\left(Q_{i,l_i}, \bigoplus_{(k,l):\,(k,l)\sqsupset(i,j)} Q_{k,l}\right) = \operatorname{Rej}\left(Q_{i,l_i}, \bigoplus_{(k,l):\,(k,l)\not\sqsubseteq(i,j)} Q_{k,l}\right).$$

In particular, $T(i,j)$ is the largest submodule of Q_{i,l_i} whose composition factors are all of the form $L_{k,l}$, with $(k,l) \sqsubseteq (i,j)$.

Proof. We give an explicit prove of this result. Recall the characterisation of the reject of injectives in Remark 1.4.1. By Theorem 2.5.8, the module $Q_{i,l_i}/T(i,j)$ has a ∇-filtration whose factors are $\nabla(i,1)$, $\nabla(i,2), \dots, \nabla(i,j-1)$. So $Q_{i,l_i}/T(i,j)$ is cogenerated by the injectives $Q_{k,l}$, with $(k,l) \sqsupset (i,j)$, that is

$$\operatorname{Rej}\left(Q_{i,l_i}, \bigoplus_{(k,l):\,(k,l)\sqsupset(i,j)} Q_{k,l}\right) \subseteq T(i,j).$$

On the other hand, all composition factors of $T(i,j)$ are of the form $L_{k,l}$, with $(k,l) \not\sqsupset (i,j)$. This proves that $T(i,j)$ is contained in $\operatorname{Rej}(Q_{i,l_i}, \bigoplus_{(k,l):\,(k,l)\sqsupset(i,j)} Q_{k,l})$.

The proof of the other equality in the statement of the lemma follows from Proposition 1.4.6 (alternatively, it can be established directly in a similar way). □

As we shall see next, the injective indecomposable modules over a RUSQ algebra can be determined from the filtrations (2.10) in Theorem 2.5.8. Furthermore, we claim that $Q_{i,j}/\nabla(i,j)$ is isomorphic to $Q_{i,j-1}$ for $1 < j \leq l_i$, and that $Q_{i,1} \cong \nabla(i,1)$.

Proposition 2.5.11. *Let $(B, \Phi, \sqsubseteq)$ be a RUSQ algebra. For every $(i,j) \in \Phi$, there are short exact sequences*

$$0 \longrightarrow \nabla(i,j) \longrightarrow Q_{i,j} \longrightarrow Q_{i,j-1} \longrightarrow 0, \tag{2.11}$$

$$0 \longrightarrow T(i,j+1) \longrightarrow T(i,1) \longrightarrow Q_{i,j} \longrightarrow 0, \tag{2.12}$$

where $Q_{i,0} := 0$. Moreover, the module $Q_{i,j}$ has a unique ∇-filtration.

Proof. By Theorem 2.5.8, we have the exact sequences

$$0 \longrightarrow T(i,j)/T(i,j+1) \longrightarrow Q_{i,l_i}/T(i,j+1) \longrightarrow Q_{i,l_i}/T(i,j) \longrightarrow 0 \quad , \tag{2.13}$$

where $T(i,l_i+1) = 0$ and $T(i,j)/T(i,j+1) \cong \nabla(i,j)$, $1 \leq j \leq l_i$. By Theorem 2.4.3, the modules $T(i,j)$, $1 \leq j \leq l_i$, have injective dimension at most one. As

Q_{i,l_i} is the injective hull of $T(i,j)$, we get that all $Q_{i,l_i}/T(i,j)$ are injective. The modules $Q_{i,l_i}/T(i,j+1)$ have a unique ∇-filtration by Theorem 2.5.8, so they are indecomposable (see Remark 2.5.9). Therefore $Q_{i,l_i}/T(i,j+1)$ is the injective hull of $\nabla(i,j)$ for every $1 \le j \le l_i$, which shows that $Q_{i,l_i}/T(i,j+1) = T(i,1)/T(i,j+1)$ is isomorphic to $Q_{i,j}$. This produces the short exact sequence (2.12) in the statement of this proposition. Now (2.13) gives the exact sequence (2.11). □

2.6 The Ringel dual of an ultra strongly quasihereditary algebra

Recall the general setup for the Ringel dual of a quasihereditary algebra summarised in Subsection 1.4.3. In this section we study the Ringel dual $\mathcal{R}(B)$ of a RUSQ algebra B. The main goal is to show that $\mathcal{R}(B)^{op}$ is also a RUSQ algebra, or, in other words, that $\mathcal{R}(B)$ is a LUSQ algebra.

Assume that $(B, \Phi, \sqsubseteq)$ is a RUSQ algebra and label the simple B-modules by (i,j), as described in (2.6). The algebra $\mathcal{R}(B)$ is quasihereditary with respect to the poset $(\Phi, \sqsubseteq^{op})$. Following Subsection 1.4.3, denote by $L'_{i,j}$, $P'_{i,j}$, $T'(i,j)$, $\nabla(i,j)'$, respectively, the simple $\mathcal{R}(B)$-modules, the projective indecomposable $\mathcal{R}(B)$-modules, etc., as naturally expected.

Let D be the standard duality for the Artin algebra $\mathcal{R}(B)$ (see Subsection 1.2.1.1). Then the standard modules over $\mathcal{R}(B)^{op}$ are the modules $D(\nabla'(i,j))$, and the indecomposable injectives are the modules $D(P'_{i,j})$. To verify that (A1) and (A2) hold for $\mathcal{R}(B)^{op}$, we need to show that

(A1*) $\nabla'(i,j)/L'_{i,j}$ is in $\mathcal{F}(\nabla')$;

(A2*) if $\nabla'(i,j)$ is simple, then $P'_{i,j}$ has a ∇'-filtration (i.e. it is a tilting module).

From the quasihereditary structure of B we can immediately deduce some properties of $\mathcal{R}(B)$.

(I) We have that $P'_{i,1} \cong T'(i,l_i)$ as $T(i,1)$ is isomorphic to Q_{i,l_i} (see the identities (1.6) in Remark 1.4.16).

(II) By applying the functor $\mathrm{Hom}_B(T,-)$ to the exact sequence (2.9) in the statement of Theorem 2.5.8, we get (using Remark 1.4.16)

$$0 \longrightarrow P'_{i,j+1} \longrightarrow P'_{i,j} \longrightarrow \Delta'(i,j) \longrightarrow 0 \quad ,$$

where $P'_{i,l_i+1} := 0$. In particular, the standard $\mathcal{R}(B)$-modules have projective dimension at most one. By the dual version of Theorem 2.4.3, this implies that the category $\mathcal{F}(\nabla')$ is closed under quotients and that $\mathcal{R}(B)$ satisfies condition (A1*). In other words, $\mathcal{R}(B)$ is a left strongly quasihereditary algebra.

(III) Using the functor $\mathrm{Hom}_B(T,-)$ and Remark 1.4.16, we get from the filtration (2.10) that the module $P'_{i,1} \cong T'(i,l_i)$ has a unique Δ'-filtration, given by

$$0 \subset P'_{i,l_i} \subset \cdots \subset P'_{i,j} \subset \cdots \subset P'_{i,1} = T'(i,l_i).$$

The quotients of this filtration are as described in (II).

Theorem 2.6.1. *Using the previous notation, $(\mathcal{R}(B), \Phi, \sqsubseteq^{op})$ is a LUSQ algebra. The following holds:*

1. $P'_{i,1} \cong T'(i,l_i)$*;*
2. *if* $1 \leq j < l_i$*, then* $T'(i,j) \cong P'_{i,1}/P'_{i,j+1}$*;*
3. $\mathcal{F}(\nabla')$ *is closed under factor modules;*
4. *for* $(i,j) \in \Phi$*, the costandard module* $\nabla'(i,j)$ *has Loewy length* j*, is uniserial, and satisfies*
$$\nabla'(i,j-1) \cong \nabla'(i,j)/L'_{i,j}.$$

Proof. Part 1 and part 3 are answered, respectively, in (I) and (II) above. Part 2 follows by applying the functor $\mathrm{Hom}_B(T,-)$ to (2.12) in Proposition 2.5.11.

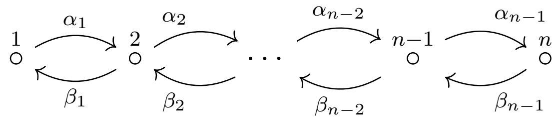
To prove part 4 and show that $(\mathcal{R}(B), \Phi, \sqsubseteq)$ is a LUSQ algebra, apply Lemma 1.4.11 to (III). This yields

$$\begin{aligned}\dim_{\mathrm{End}_{\mathcal{R}(B)}(\nabla'(i,j))} & \mathrm{Hom}_{\mathcal{R}(B)}\left(P'_{k,l}, \nabla'(i,j)\right) \\ &= \left(P'_{k,l} : \Delta'(i,j)\right) = \begin{cases} 1 & \text{if } k = i \text{ and } l \leq j, \\ 0 & \text{otherwise.}\end{cases}\end{aligned} \tag{2.14}$$

As a consequence, the composition factors of $\nabla'(i,j)$ are $L'_{i,1}, \ldots, L'_{i,j}$, with $L'_{i,j}$ having multiplicity one in $\nabla'(i,j)$. In particular, $\nabla'(i,1) \cong L'_{i,1}$, and these are all the simple costandard modules. This observation, together with part 1, implies that $\mathcal{R}(B)$ satisfies axiom (A2*). By (II), $\mathcal{R}(B)$ also satisfies (A1*), so $(\mathcal{R}(B), \Phi, \sqsubseteq^{op})$ is a LUSQ algebra. By the dual version of Proposition 2.5.5, any nonzero quotient of a costandard $\mathcal{R}(B)$-module is still a costandard module. Consequently, $\nabla'(i,j)/L'_{i,j}$ is a costandard module whose composition factors are $L'_{i,1}, \ldots, L'_{i,j-1}$. This implies that $\nabla'(i,j)/L'_{i,j} \cong \nabla'(i,j-1)$, $1 < j \leq l_i$. ∎

2.7 The ADR algebra of a certain Brauer tree algebra

Brauer tree algebras are a class of algebras of finite representation type. They include all blocks of group algebras of finite type, and also all blocks of type A Hecke algebras of finite type ([42]). In this section we determine the quiver presentation of the ADR algebra R_A of A, when A is the Brauer tree algebra KQ/I, with K an arbitrary field, Q the quiver

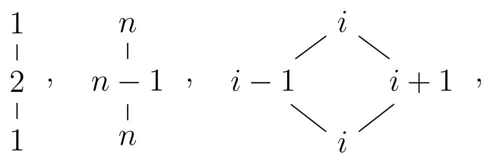


and I the admissible ideal of KQ generated by the relations

$$\alpha_{i+1}\alpha_i,\ \beta_i\beta_{i+1},\ \alpha_i\beta_i - \beta_{i+1}\alpha_{i+1},\ i = 1, \ldots, n-2.$$

The Brauer tree algebra A plays an important role in the representation theory of the symmetric group. Indeed, let Σ_m be the symmetric group on m letters. If K is a field of prime characteristic p, then any nonsimple block of $K\Sigma_m$ of finite type is Morita equivalent to the principal block of $K\Sigma_p$. Consider the algebra A defined above, with K a field of prime characteristic p and with $n = p - 1$. In this case A is a basic algebra of the principal block of $K\Sigma_p$. Moreover, the vertex i in the quiver of A may be thought as corresponding to the simple $K\Sigma_p$-module labelled by the (hook) partition $(p + 1 - i, 1^{i-1})$ of p. We refer to [54] for further details.

Since I is generated by monomial relations and by commutative relations between paths of the same length, the projective indecomposable A-modules may be represented by graphs in the following way



Denote the projective A-module corresponding to the vertex i by P_i.

According to Proposition 2.3.4, the R_A-modules $P_{i,1} = \Delta(i, 1)$ are uniserial, with Loewy length 3, and with composition factors $L_{i,1}$, $L_{i,2}$, and $L_{i,3}$, ordered from the top to the socle. Furthermore, these projectives determine all the standard R_A-modules. Consider now (for $2 \leq i \leq n-1$) the short exact sequence

$$0 \longrightarrow L_{i+1} \oplus L_{i-1} \longrightarrow P_i / \operatorname{Rad}^2 P_i \xrightarrow{\ \pi\ } L_i \longrightarrow 0 \ ,$$

and apply $\operatorname{Hom}_A(G,-)$ to it. We get the exact sequence

$$0 \longrightarrow \Delta(i+1,1) \oplus \Delta(i-1,1) \longrightarrow P_{i,2} \xrightarrow{\pi_*} \Delta(i,1) \quad ,$$

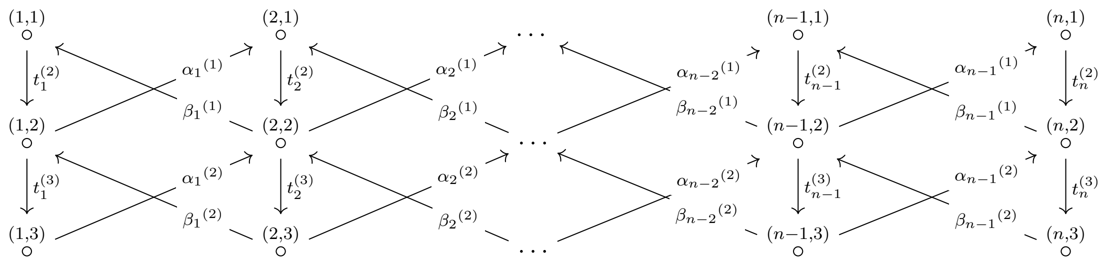
and as $\pi_* \neq 0$, we must have $\operatorname{Im} \pi_* = \Delta(i,2)$, since $\Delta(i,2)$ is the unique submodule of $\Delta(i,1)$ whose top is $L_{i,2}$. Note that this is exactly what Proposition 2.3.4 is telling us. Similarly, we get

$$0 \longrightarrow \operatorname{Hom}_A(G, \operatorname{Rad} P_i) \longrightarrow P_{i,3} \longrightarrow \Delta(i,3) \longrightarrow 0 \quad ,$$

and as $\Delta(i,3) = L_{i,3}$, it follows that $\operatorname{Hom}_A(G, \operatorname{Rad} P_i) = \operatorname{Rad} P_{i,3}$.

We wish to obtain a quiver presentation KQ'/I' for R_A. As before, denote by (i,j) the vertex of Q' corresponding to the simple R_A-module $L_{i,j}$.

Proposition 2.7.1. *The algebra R_A is isomorphic to KQ'/I', with Q' the quiver*



and I' the admissible ideal generated by the relations

$$\alpha_i^{(1)} t_i^{(2)},\ \beta_i^{(1)} t_{i+1}^{(2)},\ \alpha_i^{(2)} t_i^{(3)} - t_{i+1}^{(2)} \alpha_i^{(1)},\ \beta_i^{(2)} t_{i+1}^{(3)} - t_i^{(2)} \beta_i^{(1)},\ i = 1, \dots, n-1,$$
$${\alpha_{i+1}}^{(1)} {\alpha_i}^{(2)},\ {\beta_i}^{(1)} {\beta_{i+1}}^{(2)},\ {\alpha_i}^{(1)} {\beta_i}^{(2)} - {\beta_{i+1}}^{(1)} {\alpha_{i+1}}^{(2)},\ i = 1, \dots, n-2.$$

Proof. The vertical arrows in the quiver above correspond to the structure of the uniserial projectives $P_{i,1}$. In fact, going back to [57], one sees that the arrows

$$\overset{(i,j-1)}{\circ} \xrightarrow{t_i^{(j)}} \overset{(i,j)}{\circ}$$

correspond to the canonical epics

$$P_i / \operatorname{Rad}^j P_i \longrightarrow\!\!\!\!\!\rightarrow P_i / \operatorname{Rad}^{j-1} P_i$$

in $\operatorname{mod} A$. Let Q' be the ordinary quiver of R_A. Note that there must be exactly one arrow coming out of the vertices $(i,1)$ of Q', namely the arrow $t_i^{(2)}$ (remember that

Rad $P_{i,1}$ has simple top $L_{i,2}$ by Corollary 2.3.2). Consider now the vertices $(i,3)$ of Q'. Because P_i has Loewy length 3, it follows that

$$\operatorname{Rad} P_{i,3} = \operatorname{Hom}_A(G, \operatorname{Rad} P_i).$$

It is not difficult to show directly that Rad $P_{i,3}$ has top $L_{i-1,2} \oplus L_{i+1,2}$, $2 \leq i \leq n-1$. This also follows from Theorem A in Chapter 3. Consequently, there are exactly two arrows with source $(i,3)$ in Q' (for $2 \leq i \leq n-1$), and they must be as depicted in the quiver above. Finally, let us analyse the vertices $(i,2)$ of Q'. There are no arrows from $(i,2)$ to a vertex $(j,2)$. This is because the poset $(\Lambda, \trianglelefteq)$ is adapted to R_A (recall Definition 1.4.2), and also because $[\Delta(i,2) : L_{i,2}] = 1$. By the structure of the modules $\Delta(i,2)$, there are no arrows from $(i,2)$ to $(j,3)$ apart from the arrow $t_i^{(3)}$ already mentioned. So any other arrow in Q' having source $(i,2)$ (if any) must have sink $(j,1)$. That is, it must correspond to a map

$$L_j \hookrightarrow P_i / \operatorname{Rad}^2 P_i$$

in mod A. Conversely, any monic as the one above must correspond to an arrow from $(i,2)$ to $(j,1)$ in Q' because, by what we have seen so far, there are no alternative paths from $(i,2)$ to $(j,1)$ in Q'. As a consequence, there must be two more arrows with source $(i,2)$ (if $2 \leq i \leq n-1$), namely

$$\overset{(i,2)}{\circ} \xrightarrow{\beta_{i-1}^{(1)}} \overset{(i-1,1)}{\circ} \quad , \quad \overset{(i,2)}{\circ} \xrightarrow{\alpha_i^{(1)}} \overset{(i+1,1)}{\circ}$$

This proves that Q' coincides with the quiver in the statement of the proposition.

We have that $R_A \cong KQ'/I'$, for a certain admissible ideal I'. By the structure of $P_{i,1}$ (see Corollary 2.3.2) the paths $\alpha_i^{(1)} t_i^{(2)}$, $\beta_i^{(1)} t_{i+1}^{(2)}$ must be zero modulo I'. Besides, $\alpha_i^{(2)} t_i^{(3)} - t_{i+1}^{(2)} \alpha_i^{(1)}$ must also be zero modulo I' as the underlying diagram

$$\begin{array}{ccc} P_{i+1}/\operatorname{Rad}^2 P_{i+1} & \xrightarrow{\neq 0} & P_i/\operatorname{Rad}^3 P_i \\ \downarrow & & \downarrow \\ L_{i+1} & \hookrightarrow & P_i/\operatorname{Rad}^2 P_i \end{array}$$

commutes. Similarly, it follows that $\beta_i^{(2)} t_{i+1}^{(3)} - t_i^{(2)} \beta_i^{(1)}$ must be zero modulo I'. In a similar fashion one checks that the remaining relations in the statement of the proposition are zero modulo I'. Let $\hat{I}$ be the ideal of KQ' generated by the relations indicated in the statement of the proposition. There is an epic from $KQ'/\hat{I}$ to R_A.

It is not difficult to check that R_A has dimension $19n - 10$ as a K-vector space. It is also easy to prove by induction on n that the dimension of $KQ'/\hat{I}$ is given by the same number, which implies that $KQ'/\hat{I} \cong R_A$. $\square$

We conclude this chapter with some remarks about the algebra $R_A = KQ'/I'$, and with a result describing the ADR algebra of $K\Sigma_p$ (up to Morita equivalence) when K is a field of prime characteristic p.

Remark 2.7.2. Note that the arrows $\beta_{i-1}^{(1)}$, $\alpha_i^{(1)}$ in Q' correspond to irreducible maps in $\operatorname{mod} A$ (for the definition of irreducible morphism see [8, Section V.5]). Let M be a module in $\operatorname{mod} A$. It is clear that any irreducible map $f : X \longrightarrow Y$, with X, Y in $\operatorname{add} M$, gives rise to a morphism $f_* = \operatorname{Hom}_A(M, f)$ between projective modules in $\operatorname{mod}(\operatorname{End}_A(M)^{op})$, satisfying $\operatorname{Im} f_* \subseteq \operatorname{Rad}(\operatorname{Hom}_A(M, Y))$, $\operatorname{Im} f_* \not\subseteq \operatorname{Rad}^2(\operatorname{Hom}_A(M, Y))$.

Remark 2.7.3. Let A be as before. By Theorem 10.3 in [5] (see also Corollary 3.3.12 in Chapter 3), $\operatorname{gl.dim} R_A \leq 3$. Proposition 2 in [56] implies that $\operatorname{gl.dim} R_A \neq 2$. Hence $\operatorname{gl.dim} R_A = 3$. Moreover, Theorem B in Chapter 3 will imply that the Ringel dual of R_A is isomorphic to $(R_A)^{op}$ for every Brauer tree algebra A.

Remark 2.7.4. Let $A = KQ/I$ be a presentation of some finite-dimensional K-algebra A by a quiver and admissible relations. There is an algorithmic procedure to compute a presentation of R_A in terms of quivers and relations.

Let A be an Artin algebra and suppose that $A = \bigoplus_{j=1}^{r} I_j$ is a *block decomposition* of A, i.e. a decomposition of A into a direct sum of indecomposable ideals (so $I_j = A\xi_j$, where $\{\xi_j : j = 1, \ldots, r\}$ is a maximal set of central, pairwise orthogonal idempotents in A). Since there are no homomorphisms between (indecomposable) modules in distinct blocks, and since R_A is defined as an endomorphism algebra, then we have an isomorphism of algebras

$$R_A \cong \bigoplus_{j=1}^{r} R_{I_j},$$

which in fact corresponds to the block decomposition of R_A. Using this observation, together with Proposition 2.7.1 and the facts about the algebra of the symmetric group mentioned in the beginning of this section, we deduce the following result.

Theorem 2.7.5. *Let $A = K\Sigma_p$, where K is a field of characteristic p. Then R_A is Morita equivalent to*

$$KQ'/I' \oplus C,$$

where Q' and I' are as in Proposition 2.7.1, and C is a product of fields.

Chapter 3

Δ-semisimple filtrations, and the relationship between $\mathcal{R}(R_A)$ and $R_{A^{op}}$

3.1 Overview of the chapter

This chapter complements the investigation done in Chapter 2. We start by studying the Δ-filtrations of modules over RUSQ algebras and then specialise to the ADR algebra. Our preliminary conclusions about these algebras are then used to derive the main contributions of this chapter: we give a counterexample to a claim by Auslander–Platzeck–Todorov and prove a theorem relating the Ringel dual of the algebra R_A to the algebra $R_{A^{op}}$. We now proceed to give a more detailed description of the contents of this chapter.

In Section 3.2, we show that the RUSQ algebras satisfy the following key property: every submodule of a direct sum of standard modules is still a direct sum of standard modules. This has several consequences and, in particular, gives rise to special filtrations of Δ-good modules over the ADR algebra R_A.

Next, we describe the right minimal $\operatorname{add} G$-approximations of rigid modules in $\operatorname{mod} A$, or equivalently, the projective covers of the R_A-modules $\operatorname{Hom}_A(G, M)$, with M rigid. Recall that a module is said to be *rigid* if its radical series coincides with its socle series. We prove the following theorem in Section 3.3.

Theorem A. *Let M be a rigid module in $\operatorname{mod} A$, with Loewy length m. Then the projective cover of M in $\operatorname{mod}(A/\operatorname{Rad}^m A)$ is a right minimal $\operatorname{add} G$-approximation of M.*

This simple yet useful result, combined with the conclusions in Section 3.2, is then used to provide a counterexample to a claim by Auslander, Platzeck and Todorov in [6], about the projective resolutions of modules over the ADR algebra.

Theorem A is also a key ingredient in the proof of the central result of this chapter, Theorem B, which is concerned with the Ringel dual of the algebra R_A.

Theorem B. *Let A be an Artin algebra with Loewy length L and assume that all projective and injective indecomposable A-modules are rigid with Loewy length L. The Ringel dual of the quasihereditary algebra $(R_A, \Lambda, \trianglelefteq)$ is isomorphic to the algebra $(R_{A^{op}})^{op}$.*

The proof of Theorem B, or rather the technical results used to establish it, occupy a considerable portion of this chapter (Section 3.4). These preparatory results are of independent interest. Theorem 3.4.5 provides a complete description of the Δ-filtrations of the tilting R_A-modules in terms of the socle series of the injective indecomposable A-modules, when A is an Artin algebra whose projectives are rigid modules with Loewy length $L = \mathrm{LL}(A)$. Philosophically speaking, the technicalities in Section 3.4 are due to the fact that we are seeking to identify two algebras in a "noncanonical way".

3.2 Δ-semisimple modules and Δ-semisimple filtrations

For a quasihereditary algebra $(B, \Phi, \sqsubseteq)$, we say that a B-module is Δ-*semisimple* if it is a direct sum of standard modules. Every module M in $\mathcal{F}(\Delta)$ has some submodule N such that:

(B) N is Δ-semisimple and M/N is in $\mathcal{F}(\Delta)$.

Given a module M in $\mathcal{F}(\Delta)$, we may consider the submodules of M which are maximal with respect to property (B). The module M may have more than one such submodule (see Example 2.20 in [46]). However, according to [46], the submodules of M which are maximal with respect to (B) are unique up to isomorphism.

Suppose now that B is a RUSQ algebra. The Δ-semisimple modules over RUSQ algebras are particularly well behaved. As we will see in Corollary 3.2.3, the property of being Δ-semisimple is closed under submodules in this case. Furthermore, every module M in $\mathcal{F}(\Delta)$ has exactly one submodule D_M which is maximal with respect to property (B). The module D_M is actually the unique maximal Δ-semisimple submodule of M (with respect to inclusion). Moreover, D_M will be obtained by applying a certain hereditary preradical (as in Definition 1.3.7) to the module M. Since M/D_M still lies in $\mathcal{F}(\Delta)$, we may proceed iteratively and define the Δ-semisimple filtration (which will be unique) and the Δ-semisimple length of any module in $\mathcal{F}(\Delta)$. We shall see that the Δ-semisimple filtrations exhibit properties similar to those of a socle series.

3.2.1 Δ-semisimple modules

We are interested in submodules of Δ-good modules which are maximal with respect to property (B). As a consequence of Theorem 2.17 in [46][1], these are unique up to isomorphism.

Theorem 3.2.1 ([46, Theorem 2.17])**.** *Let $(B, \Phi, \sqsubseteq)$ be a quasihereditary algebra, and let M be in $\mathcal{F}(\Delta)$. Any two submodules of M which are maximal with respect to property (B) are isomorphic.*

We wish to study the Δ-semisimple modules over a RUSQ algebra $(B, \Phi, \sqsubseteq)$. Throughout this chapter we will adopt the notation introduced in Subsection 2.5.1. In particular, we shall assume that the set Φ is as described in (2.6) (recall that this assumption can always be made). In this subsection we prove some key properties of the Δ-semisimple modules over RUSQ algebras. Namely, we show that the property of being Δ-semisimple is closed under taking submodules.

Recall that the standard modules over a RUSQ algebra $(B, \Phi, \sqsubseteq)$ are uniserial and satisfy $\operatorname{Rad}\Delta(i,j) = \Delta(i,j+1)$. The module $\Delta(i,j)$, $(i,j) \in \Phi$, has composition factors $L_{i,j}, \ldots, L_{i,l_i}$, ordered from the top to the socle (see Proposition 2.5.6).

Lemma 3.2.2. *Let $(B, \Phi, \sqsubseteq)$ be a RUSQ algebra. Let M be in $\operatorname{mod} B$ and consider a short exact sequence*

$$0 \longrightarrow \Delta(k,l) \longrightarrow M \longrightarrow \Delta(i,j) \longrightarrow 0 \quad , \tag{3.1}$$

with $(k,l), (i,j) \in \Phi$. If $\operatorname{Soc} M \ncong \operatorname{Soc}\Delta(k,l)$, then (3.1) splits.

Proof. We prove this statement by descending induction on j. For $j = l_i$ we have $\Delta(i,j) = \Delta(i,l_i) = L_{i,l_i}$. If $\operatorname{Soc} M$ is not isomorphic to $\operatorname{Soc}\Delta(k,l)$, then the corresponding exact sequence (3.1) splits.

[1] The notion of a "Θ-semisimple subobject" introduced in this paper is stronger than ours. That is, in [46] a "Δ-semisimple submodule" would be a submodule satisfying property (B).

Suppose now that $j \leq l_i - 1$ and consider the pullback diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Delta(k,l) & \longrightarrow & N & \xrightarrow{f'} & \Delta(i,j+1) & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \iota'} & & \downarrow{\scriptstyle \iota} & & \\
0 & \longrightarrow & \Delta(k,l) & \longrightarrow & M & \xrightarrow{f} & \Delta(i,j) & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle \operatorname{coker}\iota'} & & \downarrow{\scriptstyle \operatorname{coker}\iota} & & \\
 & & & & L_{i,j} & = & L_{i,j} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \quad , \tag{3.2}$$

where $\operatorname{Soc} M \ncong \operatorname{Soc} \Delta(k,l)$ (so $\operatorname{Soc} M \cong \operatorname{Soc} \Delta(k,l) \oplus \operatorname{Soc} \Delta(i,j) \cong L_{k,l_k} \oplus L_{i,l_i}$). We wish to prove that f is a split epic.

Look at the central column of (3.2). We claim that $\operatorname{Soc} N \ncong \operatorname{Soc} \Delta(k,l)$. In fact, $\operatorname{Soc} N \cong \operatorname{Soc} \Delta(k,l)$, together with $\operatorname{Soc} M \ncong \operatorname{Soc} \Delta(k,l)$, would imply that $\operatorname{Soc} M \cong \operatorname{Soc} N \oplus L_{i,j} \cong L_{k,l_k} \oplus L_{i,j}$, which would lead to a contradiction since $j \leq l_i - 1$. Consequently, $\operatorname{Soc} N \ncong \operatorname{Soc} \Delta(k,l)$ and the short exact sequence in the top row of (3.2) splits by induction. Let μ be such that $f' \circ \mu = 1_{\Delta(i,j+1)}$, and consider the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \Delta(k,l) & \overset{\sim}{\dashrightarrow} & \operatorname{Ker} h & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Delta(i,j+1) & \xrightarrow{\iota' \circ \mu} & M & \longrightarrow & W & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \exists h} & & \\
0 & \longrightarrow & \Delta(i,j+1) & \xrightarrow{\iota} & \Delta(i,j) & \longrightarrow & L_{i,j} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \quad , \tag{3.3}$$

where $W := \operatorname{Coker}(\iota' \circ \mu)$.

Our goal is to prove that the central column of (3.3) splits. Suppose, by contradiction, that this exact sequence does not split. Then the right hand column does not split either. As a consequence, the modules W and $\operatorname{Ker} h$ have the same socle, namely $\operatorname{Soc} W \cong \operatorname{Soc} \Delta(k,l) \cong L_{k,l_k}$. Proposition 2.5.6 implies that W is in $\mathcal{F}(\Delta)$. The module W has exactly one composition factor of the form L_{x,l_x} (as $j \leq l_i - 1$). This means that W must be a standard module (see Proposition 2.5.6). Hence $(i,j) = (k, l-1)$. Since $(k, l-1) \not\sqsubseteq (k,l)$, part 1 of Lemma 1.4.5 implies that the central column of (3.3) splits, which contradicts our assumption. This finishes the proof of the lemma. □

We now use the previous result to give a characterisation of the Δ-semisimple modules over a RUSQ algebra.

Corollary 3.2.3. *Let $(B, \Phi, \sqsubseteq)$ be a RUSQ algebra and let M be in $\mathcal{F}(\Delta)$. Then M is Δ-semisimple if and only if the number of simple summands of* $\operatorname{Soc} M$ *coincides with the number of factors in a Δ-filtration of M. Moreover, any submodule of a Δ-semisimple module is still Δ-semisimple.*

Proof. Let M be in $\mathcal{F}(\Delta)$. Denote by $\mathcal{P}(M)$ the following assertion: "the number of simple summands of $\operatorname{Soc} M$ coincides with the number of factors in a Δ-filtration of M". By parts 4 and 5 of Proposition 2.5.6, $\mathcal{P}(M)$ is true if and only if the composition factors of M of type L_{x,l_x} are exactly the summands of its socle. From this equivalence, it is easy to see that the truth of $\mathcal{P}(M)$ implies the truth of $\mathcal{P}(N)$ for $N \subseteq M$ (note that the inclusion of N in M induces a monic from $N/\operatorname{Soc} N$ to $M/\operatorname{Soc} M$ – see Example 1.3.8 and Lemma 1.3.9).

If M is a Δ-semisimple module then $\mathcal{P}(M)$ is clearly true. Suppose now that $\mathcal{P}(M)$ holds for $M \in \mathcal{F}(\Delta)$. We wish to show that M is Δ-semisimple. We prove this by induction on the number z of factors in a Δ-filtration of M. If $z = 1$ the result is obvious. Suppose now that $z \geq 2$, and consider a short exact sequence

$$0 \longrightarrow \operatorname{Ker} f \longrightarrow M \xrightarrow{f} \Delta(i,j) \longrightarrow 0 \quad ,$$

where $\operatorname{Ker} f \in \mathcal{F}(\Delta)$. Since $\mathcal{P}(M)$ holds, then $\mathcal{P}(\operatorname{Ker} f)$ also holds by the previous remark. By induction, $\operatorname{Ker} f$ must be a Δ-semisimple module. Consider now the pushout diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Ker} f & \xrightarrow{\ker f} & M & \xrightarrow{f} & \Delta(i,j) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} & & \| & & \\ 0 & \longrightarrow & \Delta(k,l) & \xrightarrow{h} & N & \longrightarrow & \Delta(i,j) & \longrightarrow & 0 \end{array} \quad , \tag{3.4}$$

where π is a split epic mapping the Δ-semisimple module $\operatorname{Ker} f$ onto a standard module $\Delta(k,l)$. Observe that $\operatorname{Soc} M \cong \operatorname{Soc}(\operatorname{Ker} f) \oplus \operatorname{Soc} \Delta(i,j)$, as the composition factors of M of type L_{x,l_x} are exactly the summands of its socle. Recall that $\operatorname{Soc}(-)$ is a left exact functor (see Lemma 1.3.9). By applying $\operatorname{Soc}(-)$ to (3.4), we get a new diagram where the top row remains exact, and consequently the bottom row also remains exact. Thus $\operatorname{Soc} N \not\supseteq \operatorname{Soc} \Delta(k,l)$, and, by Lemma 3.2.2, the bottom row of (3.4) splits. So there is an epic ρ satisfying $\rho \circ h = 1_{\Delta(k,l)}$. Since π is a split epic, there is a monic μ such that $\pi \circ \mu = 1_{\Delta(k,l)}$. Notice that $(\ker f) \circ \mu$ is a split monic, as

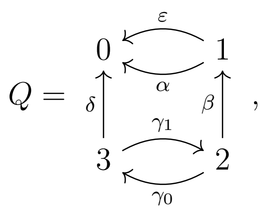
$$\rho \circ \pi' \circ (\ker f) \circ \mu = \rho \circ h \circ \pi \circ \mu = 1_{\Delta(k,l)}.$$

So $M \cong \Delta(k,l) \oplus M'$, for some module M' in $\operatorname{mod} B$. The module M' lies in $\mathcal{F}(\Delta)$ since this category is closed under submodules. In fact, $\mathcal{P}(M')$ is true by the observation in the beginning of the proof. By induction, the module M' must be Δ-semisimple. Therefore M is Δ-semisimple as well. This proves the first claim in the statement of the corollary.

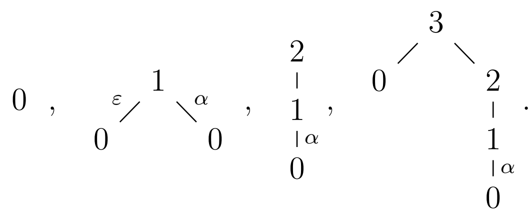
Let now N be a submodule of a Δ-semisimple module M. Then $\mathcal{P}(M)$ is true, which implies that $\mathcal{P}(N)$ holds. The first claim implies that N is Δ-semisimple. □

In the next subsection we are going to show that the Δ-good modules over a RUSQ algebra have a unique maximal Δ-semisimple submodule. First, we check that arbitrary quasihereditary algebras do not possess this property.

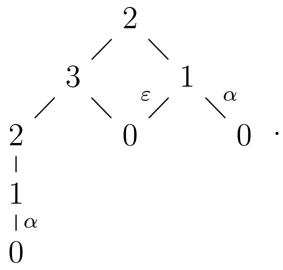
Example 3.2.4. Consider the quiver



and the bound quiver algebra $B = KQ/I$, where I is the ideal generated by the elements $\varepsilon\beta - \delta\gamma_0$ and $\gamma_0\gamma_1$. It is easy to check that B is quasihereditary with respect to the labelling poset $0 < 1 < 2 < 3$. The modules



are the corresponding standard B-modules. The projective cover P_2 of the simple module with label 2 has the following structure



The modules $\Delta(1) \oplus \Delta(2)$ and $\Delta(3) \oplus \Delta(0)$ are both maximal Δ-semisimple submodules of P_2. The quotient of P_2 by each of these submodules does not belong to $\mathcal{F}(\Delta)$, i.e. none of these submodules of P_2 satisfies property (B).

3.2.2 The preradical δ and Δ-semisimple filtrations

Let $(B, \Phi, \sqsubseteq)$ be an arbitrary quasihereditary algebra. As pointed out in the previous subsection, the submodules of a module M in $\mathcal{F}(\Delta)$ which are maximal with respect to property (B) are all isomorphic, but they are not necessarily unique. We have also seen that a module M in $\mathcal{F}(\Delta)$ may have more than one maximal Δ-semisimple submodule with respect to inclusion (Example 3.2.4). We shall prove that both these maximal submodules are unique and actually coincide when the underlying algebra is a RUSQ algebra. For this, we use the general theory of preradicals introduced in Section 1.3.

Recall the definition of a hereditary class (Definition 1.3.13).

Lemma 3.2.5. *Let $(B, \Phi, \sqsubseteq)$ be a RUSQ algebra. The corresponding set Δ of standard B-modules is a hereditary class in* $\operatorname{mod} B$*. In particular,* $\operatorname{Tr}(\Delta, -)$ *is a hereditary preradical in* $\operatorname{mod} B$.

Proof. Let N be a submodule of a module in $\operatorname{add}\Delta$, so N is contained in some Δ-semisimple module M. By Corollary 3.2.3, N is still Δ-semisimple, so it is trivially generated by Δ. Hence the set Δ is hereditary. Lemma 1.3.14 implies that $\operatorname{Tr}(\Delta, -)$ is a hereditary preradical in $\operatorname{mod} B$. □

From now onwards we shall denote the functor $\operatorname{Tr}(\Delta, -)$ by δ.

Definition 3.2.6. For a RUSQ algebra $(B, \Phi, \sqsubseteq)$, let δ be the hereditary preradical $\operatorname{Tr}(\Delta, -)$ in $\operatorname{mod} B$.

Next, we give a description of the submodule $\delta(M)$ of a module $M \in \mathcal{F}(\Delta)$.

Proposition 3.2.7. *Let $(B, \Phi, \sqsubseteq)$ be a RUSQ algebra, and let $M \in \mathcal{F}(\Delta)$. Then $\delta(M)$ is the largest Δ-semisimple submodule of M. Furthermore, $M/\delta(M)$ lies in $\mathcal{F}(\Delta)$. In particular, $\delta(M)$ is the largest submodule of M satisfying property (B).*

Proof. By the definition of $\mathrm{Tr}(\Delta, -)$, there is an epic f from a Δ-semisimple module M' to $\delta(M)$. Note that $\delta(M)$ is in $\mathcal{F}(\Delta)$, since this category is closed under submodules. By parts 5 and 3 of Lemma 1.4.8, f must be a split epic. Hence $\delta(M)$ is Δ-semisimple. By the definition of $\mathrm{Tr}(\Delta, -)$ it is clear that every Δ-semisimple submodule of M must be contained in $\delta(M)$. This shows that $\delta(M)$ is the largest Δ-semisimple submodule of $M \in \mathcal{F}(\Delta)$.

To conclude this proof it is enough to show that $M/\delta(M)$ lies in $\mathcal{F}(\Delta)$. We start by proving that this holds for the injective modules $Q_{i,l_i} = T(i,1)$ (recall Proposition 2.5.6). Note that $\Delta(i,1) \subseteq \delta(Q_{i,l_i})$, as $\Delta(i,1)$ is a submodule of $T(i,1)$. Since Q_{i,l_i} has simple socle L_{i,l_i}, then $\delta(Q_{i,l_i})$ has to be isomorphic to some standard module $\Delta(i,j)$. But then we must have $\Delta(i,1) = \delta(Q_{i,l_i})$, and consequently $Q_{i,l_i}/\delta(Q_{i,l_i}) = T(i,1)/\Delta(i,1)$ is in $\mathcal{F}(\Delta)$. Let now Q be a finite direct sum of injective modules of type Q_{i,l_i}. The module $Q/\delta(Q)$ still lies in $\mathcal{F}(\Delta)$ because preradicals preserve finite direct sums (see part 3 of Lemma 1.3.3). Consider now M in $\mathcal{F}(\Delta)$. By Proposition 2.5.6, the injective hull $q_0 : M \longrightarrow Q_0(M)$ of $M \in \mathcal{F}(\Delta)$ is such that $Q_0(M)$ is a direct sum of injectives of type Q_{i,l_i}. By part 3 of Lemma 1.3.9, q_0 gives rise to a monic $M/\delta(M) \longrightarrow Q_0(M)/\delta(Q_0(M))$, and by our previous observation $Q_0(M)/\delta(Q_0(M))$ lies in $\mathcal{F}(\Delta)$. As $\mathcal{F}(\Delta)$ is closed under submodules, the module $M/\delta(M)$ belongs to $\mathcal{F}(\Delta)$. □

Example 3.2.8. Note that for an arbitrary quasihereditary algebra the modules $\delta(M)$, $M \in \mathcal{F}(\Delta)$, are not usually Δ-semisimple (not even Δ-good). Indeed, for the algebra in Example 3.2.4, we have $\delta(P_2) = \mathrm{Tr}(\Delta, P_2) = \mathrm{Rad}\, P_2$, which is not Δ-semisimple.

Note that $\delta(M) \neq 0$ for every nonzero module M in $\mathrm{mod}\, B$ as

$$\mathrm{Soc}\, M \subseteq \mathrm{Tr}(\Delta, M) = \delta(M).$$

In fact, we have $\mathrm{Soc}\, M = \mathrm{Soc}\, \delta(M)$. We may construct the preradicals δ_m in $\mathrm{mod}\, B$ defined recursively in Subsection 1.3.3. The odd numbered parts of Lemma 1.3.18, as well as Lemmas 1.3.19 and 1.3.21, hold for the preradicals δ_m. In particular, δ_m is a hereditary preradical for every $m \in \mathbb{Z}_{\geq 0}$.

Lemma 3.2.9. *Let* $(B, \Phi, \sqsubseteq)$ *be a RUSQ algebra. If* M *is in* $\mathcal{F}(\Delta)$ *then so is* $M/\delta_m(M)$*, for any* $m \geq 0$*.*

Proof. By Proposition 3.2.7, the claim holds for $m = 1$. Suppose $m \geq 2$. Then

$$\begin{aligned} M/\delta_m(M) &\cong (M/\delta(M)) / (\delta_{m-1} \bullet \delta(M)/\delta(M)) \\ &= (M/\delta(M)) / (\delta_{m-1}(M/\delta(M))), \end{aligned}$$

so by induction $M/\delta_m(M)$ belongs to $\mathcal{F}(\Delta)$. $\square$

Given a module M in $\mathcal{F}(\Delta)$, we may consider the filtration

$$0 \subset \delta(M) \subset \cdots \subset \delta_m(M) = M, \tag{3.5}$$

where $m = l^{(\delta,\bullet)}(M)$ is as defined in Lemma 1.3.19. The factors of this filtration are Δ-semisimple: by Lemma 3.2.9 and Proposition 3.2.7 the modules $\delta_i(M)/\delta_{i-1}(M) = \delta(M/\delta_{i-1}(M))$ are Δ-semisimple. We call (3.5) the Δ-*semisimple filtration* of $M \in \mathcal{F}(\Delta)$.

Definition 3.2.10. The Δ-*semisimple length* of a module M in $\mathcal{F}(\Delta)$, denoted by $\Delta.\operatorname{ssl} M$, is the length of the Δ-semisimple filtration of M, i.e. it is given by the number $l^{(\delta,\bullet)}(M)$ (as in Lemma 1.3.19).

Lemma 3.2.11. *Let* $(B, \Phi, \sqsubseteq)$ *be a RUSQ algebra, and let* M *be in* $\mathcal{F}(\Delta)$*, with* $\Delta.\operatorname{ssl} M = m \geq 2$*. Suppose that* $2 \leq i \leq m$*, and let* π *be a split epic mapping the* Δ*-semisimple module* $\delta_i(M)/\delta_{i-1}(M)$ *onto a summand* $\Delta(k,l)$*. Then the epic*

$$\delta_i(M)/\delta_{i-2}(M) \twoheadrightarrow \delta_i(M)/\delta_{i-1}(M) \xrightarrow{\pi} \Delta(k,l)$$

does not split.

Proof. Denote the canonical epic $\delta_i(M)/\delta_{i-2}(M) \longrightarrow \delta_i(M)/\delta_{i-1}(M)$ by ϖ. Suppose, by contradiction, that $\pi \circ \varpi$ splits. Then $\delta_i(M)/\delta_{i-2}(M) \cong \operatorname{Ker}(\pi \circ \varpi) \oplus \Delta(k,l)$. Note that $\operatorname{Ker}\varpi \subseteq \operatorname{Ker}(\pi \circ \varpi)$, and that $\operatorname{Ker}\varpi \cong \delta_{i-1}(M)/\delta_{i-2}(M)$. As a consequence, there is a monic from $\operatorname{Ker}\varpi \oplus \Delta(k,l)$ to $\delta_i(M)/\delta_{i-2}(M)$, so $\operatorname{Soc}(\delta_{i-1}(M)/\delta_{i-2}(M)) \oplus L_{k,l_k}$ can be embedded in $\operatorname{Soc}(\delta_i(M)/\delta_{i-2}(M))$. It is easy to check that $\delta_{i-1}(M)/\delta_{i-2}(M) = \delta(\delta_i(M)/\delta_{i-2}(M))$ (see Lemma 1.3.21). Therefore $\operatorname{Soc}(\delta_i(M)/\delta_{i-2}(M)) = \operatorname{Soc}(\delta_{i-1}(M)/\delta_{i-2}(M))$, which leads to a contradiction. $\square$

Results similar to Lemmas 3.2.9 and 3.2.11 often hold in more general situations involving sequences of hereditary preradicals satisfying the conditions of Lemma 1.3.21. The functors $\operatorname{Soc}_i(-)$, $i \in \mathbb{Z}_{\geq 0}$, are the classic example of such a sequence of preradicals. For instructive purposes it is often useful to keep in mind this analogy between the functors δ_i and $\operatorname{Soc}_i(-)$.

3.2.3 Δ-semisimple filtrations of modules over the ADR algebra

The ADR algebra of an Artin algebra A, $R = (R_A, \Lambda, \trianglelefteq)$, is our prototype of a RUSQ algebra. We now prove some results specific to the Δ-semisimple filtrations of Δ-good modules over the ADR algebra. Throughout this subsection the underlying quasihereditary algebra will be $(R, \Lambda, \trianglelefteq)$, where the poset $(\Lambda, \trianglelefteq)$ is as defined in (2.2) and (2.3). For the proof of the next results it is useful to remember that the left exact functor $\mathrm{Hom}_A(G, -)$ is fully faithful.

Lemma 3.2.12. *Let M_1 and M_2 be in $\mathrm{mod}\, A$, with $M_1 \subseteq M_2$. There is a canonical embedding*

$$\mathrm{Hom}_A(G, M_2) / \mathrm{Hom}_A(G, M_1) \overset{\iota}{\hookrightarrow} \mathrm{Hom}_A(G, M_2/M_1)$$

and the induced morphisms

$$\mathrm{Hom}_R(P_{i,l_i}, \mathrm{Hom}_A(G, M_2) / \mathrm{Hom}_A(G, M_1)) \overset{\iota_*}{\hookrightarrow} \mathrm{Hom}_R(P_{i,l_i}, \mathrm{Hom}_A(G, M_2/M_1)) \quad ,$$

$$\mathrm{Hom}_R(\mathrm{Hom}_A(G, M_2/M_1), Q_{i,l_i}) \overset{\iota^*}{\twoheadrightarrow} \mathrm{Hom}_R(\mathrm{Hom}_A(G, M_2) / \mathrm{Hom}_A(G, M_1), Q_{i,l_i})$$

are isomorphisms.

Proof. The functor $\mathrm{Hom}_A(G, -)$ is left exact. Thus, it maps the canonical epic $\pi : M_2 \longrightarrow M_2/M_1$ to the morphism π_*, which factors as

$$\begin{array}{ccc} \mathrm{Hom}_A(G, M_2) & \xrightarrow{\quad\pi_*\quad} & \mathrm{Hom}_A(G, M_2/M_1) \\ & {\scriptstyle\varpi}\searrow\!\!\!\!\twoheadrightarrow \quad \hookrightarrow\!\!\nearrow{\scriptstyle\iota} & \\ & \mathrm{Hom}_A(G, M_2) / \mathrm{Hom}_A(G, M_1) & \end{array} .$$

Consider the monic ι_* obtained by applying the functor $\mathrm{Hom}_R(P_{i,l_i}, -)$ to ι. Let f_* be in $\mathrm{Hom}_R(P_{i,l_i}, \mathrm{Hom}_A(G, M_2/M_1))$. Then $f_* = \mathrm{Hom}_A(G, f)$, for a map $f : P_i \longrightarrow M_2/M_1$ in $\mathrm{mod}\, A$. Since P_i is projective then $f = \pi \circ t$ for some $t : P_i \longrightarrow M_2$. So $f_* = \pi_* \circ t_* = \iota \circ \varpi \circ t_* = \iota_*(\varpi \circ t_*)$, where $t_* = \mathrm{Hom}_A(G, t)$. This shows that ι_* is surjective, hence it is an isomorphism. The proof that ι^* is an isomorphism is analogous. □

Let M_1 and M_2 be in $\mathrm{mod}\, A$, with $M_1 \subseteq M_2$. We shall regard the canonical embedding in Lemma 3.2.12,

$$\mathrm{Hom}_A(G, M_2) / \mathrm{Hom}_A(G, M_1) \overset{\iota}{\hookrightarrow} \mathrm{Hom}_A(G, M_2/M_1) \quad ,$$

as an inclusion of R-modules. According to Lemma 2.3.6, the module $\operatorname{Hom}_A(G, M)$ lies in $\mathcal{F}(\Delta)$ for every M in $\operatorname{mod} A$. Since the category $\mathcal{F}(\Delta)$ is closed under submodules then both $\operatorname{Hom}_A(G, M_2) / \operatorname{Hom}_A(G, M_1)$ and $\operatorname{Hom}_A(G, M_2/M_1)$ are Δ-good modules. Lemma 3.2.12 is hinting at a close relation between the Δ-filtrations of the modules $\operatorname{Hom}_A(G, M_2) / \operatorname{Hom}_A(G, M_1)$ and $\operatorname{Hom}_A(G, M_2/M_1)$. We spell out this idea below.

Corollary 3.2.13. *Let M_1 and M_2 be in $\operatorname{mod} A$, with $M_1 \subseteq M_2$. Write $M = \operatorname{Hom}_A(G, M_2/M_1)$ and $M' = \operatorname{Hom}_A(G, M_2) / \operatorname{Hom}_A(G, M_1)$. All the composition factors of M of type L_{i,l_i} appear as composition factors of its submodule M'. In particular, M and M' have the same number of composition factors of type L_{i,l_i}. Moreover, M and M' lie in $\mathcal{F}(\Delta)$, $\operatorname{Soc} M = \operatorname{Soc} M'$, and the modules M and M' are filtered by the same number of standard modules.*

Proof. Fix $1 \leq i \leq n$. By Lemma 3.2.12, all the composition factors of M isomorphic to L_{i,l_i} appear as composition factors of its submodule M'. As M lies in $\mathcal{F}(\Delta)$ then, by Proposition 2.5.6, $\operatorname{Soc} M$ is a direct sum of simples of type L_{i,l_i}. Thus $\operatorname{Soc} M' = \operatorname{Soc} M$. Part 4 of Proposition 2.5.6 implies that the Δ-filtrations of M and M' have the same number of factors. □

As we shall see next, the socle series of an A-module M gives rise to the Δ-semisimple filtration of $\operatorname{Hom}_A(G, M)$ in $\mathcal{F}(\Delta)$.

Lemma 3.2.14. *Let M be in $\operatorname{mod} A$. Then*

$$\operatorname{Hom}_A(G, \operatorname{Soc}_j M) = \operatorname{Tr}\left(\bigoplus_{(k,l):\, l \leq j} P_{k,l}, \operatorname{Hom}_A(G, M) \right). \tag{3.6}$$

Moreover, if $\operatorname{Soc}_j M / \operatorname{Soc}_{j-1} M = \bigoplus_{\theta \in \Theta} L_{x_\theta}$, then

$$\operatorname{Hom}_A(G, \operatorname{Soc}_j M) / \operatorname{Hom}_A(G, \operatorname{Soc}_{j-1} M) = \bigoplus_{\theta \in \Theta} \Delta(x_\theta, j).$$

Proof. By Lemma 2.3.3, $\operatorname{Hom}_A(G, \operatorname{Soc}_j M)$ is generated by projectives $P_{k,l}$ satisfying $l \leq j$. This proves one of the inclusions in (3.6). Consider now an arbitrary morphism $f_* : P_{k,l} \longrightarrow \operatorname{Hom}_A(G, M)$, with $l \leq j$. Note that $f_* = \operatorname{Hom}_A(G, f)$ for a certain map $f : P_k / \operatorname{Rad}^l P_k \longrightarrow M$. Clearly, $\operatorname{Im} f \subseteq \operatorname{Soc}_j M$. But then

$$\operatorname{Im} f_* \subseteq \operatorname{Hom}_A(G, \operatorname{Im} f) \subseteq \operatorname{Hom}_A(G, \operatorname{Soc}_j M).$$

As f_* was chosen arbitrarily, the other inclusion follows. This proves identity (3.6).

To prove the second claim in the statement of the lemma, set

$$M' := \operatorname{Hom}_A(G, \operatorname{Soc}_j M) / \operatorname{Hom}_A(G, \operatorname{Soc}_{j-1} M),$$

and assume that $\operatorname{Soc}_j M / \operatorname{Soc}_{j-1} M$ is isomorphic to $\bigoplus_{\theta\in\Theta} L_{x_\theta}$. Lemma 3.2.12 and Corollary 3.2.13 imply that M' is contained in

$$\operatorname{Hom}_A(G, \operatorname{Soc}_j M / \operatorname{Soc}_{j-1} M) = \bigoplus_{\theta\in\Theta} \operatorname{Hom}_A(G, L_{x_\theta}) = \bigoplus_{\theta\in\Theta} \Delta(x_\theta, 1)$$

and that these modules have the same socle. By Corollary 3.2.3, M' is Δ-semisimple. Finally, by the identity (3.6) (applied to j and $j-1$), the module M' must be generated by projectives of type $P_{i,j}$. This proves the second assertion of the lemma. □

Lemmas 3.2.14 and 3.2.15 are very useful to compute examples. For the proof of the next result, recall the characterisation of the preradical δ in Subsection 3.2.2, namely Proposition 3.2.7 and Lemma 3.2.9.

Lemma 3.2.15. *Let M be in* $\operatorname{mod} A$. *The socle series of M induces the Δ-semisimple filtration of* $\operatorname{Hom}_A(G, M)$. *Formally,*

$$\delta_m(\operatorname{Hom}_A(G, M)) = \operatorname{Hom}_A(G, \operatorname{Soc}_m M),$$

for all $m \in \mathbb{Z}_{\geq 0}$. In particular, $\Delta.\operatorname{ssl}(\operatorname{Hom}_A(G, M)) = \operatorname{LL}(M)$.

Proof. For m satisfying $1 \leq m \leq \operatorname{LL}(M)$ we prove the claim by induction on m, starting with $m = 1$. Note that $\operatorname{Hom}_A(G, \operatorname{Soc} M)$ is a direct sum of standard modules of type $\operatorname{Hom}_A(G, L_i) = \Delta(i, 1)$, so $\operatorname{Hom}_A(G, \operatorname{Soc} M) \subseteq \delta(\operatorname{Hom}_A(G, M))$. Since the functor $\operatorname{Hom}_A(G, -)$ preserves injective hulls (see Lemma 2.4.4), the modules $\operatorname{Hom}_A(G, \operatorname{Soc} M)$ and $\operatorname{Hom}_A(G, M)$ have the same socle. Hence the previous inclusion must be an equality.

Suppose now that $2 \leq m \leq \operatorname{LL}(M)$, and set

$$Z_1 := \operatorname{Hom}_A(G, M) / \operatorname{Hom}_A(G, \operatorname{Soc}_{m-1} M)$$
$$Z_2 := \operatorname{Hom}_A(G, \operatorname{Soc}_m M) / \operatorname{Hom}_A(G, \operatorname{Soc}_{m-1} M).$$

Since $\operatorname{Hom}_A(G, -)$ preserves injective hulls, the modules $\operatorname{Hom}_A(G, M/\operatorname{Soc}_{m-1} M)$ and $\operatorname{Hom}_A(G, \operatorname{Soc}_m M/\operatorname{Soc}_{m-1} M)$ have the same socle. But then, by Corollary 3.2.13, Z_1 and Z_2 have the same socle. Moreover, Z_1 belongs to $\mathcal{F}(\Delta)$. By Lemma 3.2.14, Z_2 must be contained in $\delta(Z_1)$. So both $\delta(Z_1)$ and Z_2 are Δ-semisimple modules with the same socle. By Corollary 3.2.13, Z_1/Z_2 must be in $\mathcal{F}(\Delta)$. Since $\mathcal{F}(\Delta)$

is closed under submodules, then $\delta(Z_1)/Z_2$ is in $\mathcal{F}(\Delta)$. We must have $\delta(Z_1)/Z_2 = 0$, otherwise this factor module would have some composition factor of type L_{i,l_i}. By induction, we may suppose that $\delta_{m-1}(\mathrm{Hom}_A(G, M)) = \mathrm{Hom}_A(G, \mathrm{Soc}_{m-1} M)$. Then, the identity $Z_2 = \delta(Z_1)$ translates to

$$\mathrm{Hom}_A(G, \mathrm{Soc}_m M)/\delta_{m-1}(\mathrm{Hom}_A(G, M)) = \delta_m(\mathrm{Hom}_A(G, M))/\delta_{m-1}(\mathrm{Hom}_A(G, M)).$$

This implies that $\delta_m(\mathrm{Hom}_A(G, M)) = \mathrm{Hom}_A(G, \mathrm{Soc}_m M)$, $1 \leq m \leq \mathrm{LL}(M)$. The same identity holds trivially for $m = 0$ and for $m \geq \mathrm{LL}(M)$. □

3.3 Projective covers of modules over the ADR algebra

We would like to determine the projective covers of modules over the ADR algebra R_A of A. For a module M in $\mathrm{mod}\, A$, the projective cover p_* of $\mathrm{Hom}_A(G, M)$ in $\mathrm{mod}\, R_A$ is the image of an epic p, with domain in $\mathrm{add}\, G$, through the functor $\mathrm{Hom}_A(G, -)$. The morphism p is a special kind of map: it is the right minimal $\mathrm{add}\, G$-approximation of M in $\mathrm{mod}\, A$.

The problem of finding approximations is hard in general. However, as we shall see in Theorem A, it is very easy to compute right $\mathrm{add}\, G$-approximations of rigid modules.

Theorem A (or rather consequences of this result – Corollary 3.3.1 and Proposition 3.3.2) will play an important role in the proof of Theorem B, and it will also be very useful when dealing with examples.

In Subsection 3.3.2, we will use Corollary 3.3.1 and Proposition 3.3.2 to give a counterexample to a claim by Auslander, Platzeck and Todorov ([6]) about the projective resolutions of modules over the ADR algebra.

3.3.1 Theorem A

Recall the definition of right $\mathcal{X}$-approximation and of right minimal morphism, introduced in Subsection 1.2.2. By Proposition 1.2.4, the right $\mathrm{add}\, G$-approximations of a module M in $\mathrm{mod}\, A$ are in bijection with epics in $\mathrm{mod}\, R_A$,

$$\mathrm{Hom}_A(G, X) \twoheadrightarrow \mathrm{Hom}_A(G, M),$$

where $X \in \operatorname{add} G$. This bijection restricts to a one-to-one correspondence between right minimal $\operatorname{add} G$-approximations in $\operatorname{mod} A$ and projective covers in $\operatorname{mod} R$. Since G is a generator, the functor $\operatorname{Hom}_A(G, -)$ is particularly well behaved: it is fully faithful and it is such that the projective cover of a module M in $\operatorname{mod} A$ factors through its $\operatorname{add} G$-approximation. The latter statement implies that every right $\operatorname{add} G$-approximation is an epimorphism.

Theorem A. *Let M be a rigid module in $\operatorname{mod} A$ such that $\operatorname{LL}(M) = m$. The projective cover of M in $\operatorname{mod}(A/\operatorname{Rad}^m A)$ is a right minimal $\operatorname{add} G$-approximation of M.*

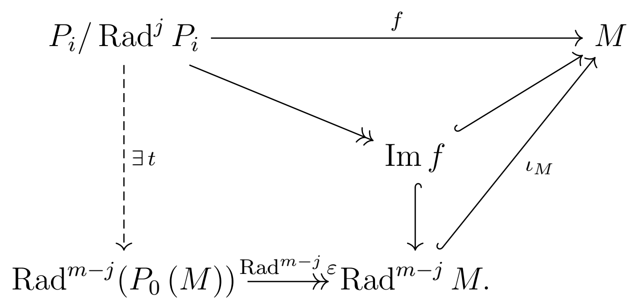
Proof. Let M be a rigid module with Loewy length m. Consider the projective cover of M as an $(A/\operatorname{Rad}^m A)$-module,

$$\varepsilon : P_0(M) \longrightarrow M.$$

We want to prove that ε is a right minimal $\operatorname{add} G$-approximation. By definition, ε is a right minimal morphism, so it is enough to prove that every map $f : P_i/\operatorname{Rad}^j P_i \longrightarrow M$, with $(i, j) \in \Lambda$, factors through ε. Note that this holds for $j \geq m$, as ε is an epic in $\operatorname{mod}(A/\operatorname{Rad}^j A)$ and $P_i/\operatorname{Rad}^j P_i$ is a projective $(A/\operatorname{Rad}^j A)$-module. So suppose that $j < m$. Then

$$\operatorname{Im} f \subseteq \operatorname{Soc}_j M = \operatorname{Rad}^{m-j} M,$$

using that M is rigid. Observe that both $\operatorname{Rad}^{m-j} M$ and $\operatorname{Rad}^{m-j}(P_0(M))$ are annihilated by $\operatorname{Rad}^j A$, i.e. they lie in $\operatorname{mod}(A/\operatorname{Rad}^j A)$. Now note that the functor $\operatorname{Rad}^{m-j}(-)$ preserves epics. This can be seen directly, or can be deduced by looking at Example 1.3.8 and Remark 1.3.10, recalling that the composition of cohereditary preradicals is still a cohereditary preradical. Therefore we have the diagram



,

where the map t exists because $P_i/\operatorname{Rad}^j P_i$ is a projective in $\operatorname{mod}(A/\operatorname{Rad}^j A)$. Thus

$$f = \iota_M \circ (\operatorname{Rad}^{m-j} \varepsilon) \circ t = \varepsilon \circ \iota_{P_0(M)} \circ t,$$

where $\iota_{P_0(M)}$ denotes the inclusion of $\operatorname{Rad}^{m-j}(P_0(M))$ in $P_0(M)$. □

As an immediate consequence of Theorem A, we get the following result.

Corollary 3.3.1. *Let M be a rigid module in* $\operatorname{mod} A$ *with* $\operatorname{LL}(M) = m$. *Suppose that ε is the projective cover of M in* $\operatorname{mod}(A/\operatorname{Rad}^m A)$. *Then* $\operatorname{Hom}_A(G, \varepsilon)$ *is the projective cover of* $\operatorname{Hom}_A(G, M)$ *in* $\operatorname{mod} R_A$.

The simple modules over the ADR algebra R_A are "linked to each other" in a neat way. When all projective indecomposable modules are rigid then the 'glueing' of the simple modules (and of the standard modules) is even nicer.

Proposition 3.3.2. *Let (i, j) and (k, l) be in Λ. Then* $\operatorname{Ext}^1_{R_A}(L_{i,j}, L_{k,l}) \neq 0$ *implies that either $(k, l) = (i, j + 1)$ or $l \leq j - 1$. If the A-module $P_i/\operatorname{Rad}^j P_i$ is rigid then* $\operatorname{Ext}^1_{R_A}(L_{i,j}, L_{k,l}) \neq 0$ *implies that either $(k, l) = (i, j + 1)$ or $l = j - 1$. In particular, the latter statement holds when all the projective indecomposable A-modules are rigid.*

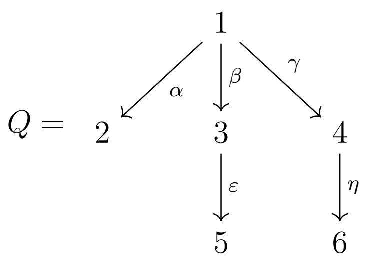
Proof. Recall that $\operatorname{Ext}^1_{R_A}(L_{i,j}, L_{k,l}) \neq 0$ if and only if the simple module $L_{k,l}$ is a summand of $\operatorname{Rad} P_{i,j}/\operatorname{Rad}^2 P_{i,j}$ (see Proposition 1.2.3). The short exact sequence (2.4) in Proposition 2.3.4 gives rise to the exact sequence

$$0 \longrightarrow \operatorname{Hom}_A\left(G, \operatorname{Rad} P_i/\operatorname{Rad}^j P_i\right) \longrightarrow \operatorname{Rad} P_{i,j} \longrightarrow \operatorname{Rad} \Delta(i, j) \longrightarrow 0,$$

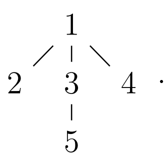
where $\operatorname{Rad} \Delta(i, j) = \Delta(i, j + 1)$. So if $L_{k,l}$ is a summand of the top of $\operatorname{Rad} P_{i,j}$ then either $(k, l) = (i, j+1)$ or $L_{k,l}$ is a summand of the top of $\operatorname{Hom}_A\left(G, \operatorname{Rad} P_i/\operatorname{Rad}^j P_i\right)$. In the latter case, we must have $l \leq j - 1$ by Lemma 2.3.3.

If $P_i/\operatorname{Rad}^j P_i$ is rigid, then $\operatorname{Rad} P_i/\operatorname{Rad}^j P_i$ is also rigid. In this case, Corollary 3.3.1 implies that the summands of the top of $\operatorname{Hom}_A\left(G, \operatorname{Rad} P_i/\operatorname{Rad}^j P_i\right)$ are of type $L_{k,j-1}$. ∎

Example 3.3.3. Consider the quiver



and the path algebra $A = KQ$. Let M be the A-module P_1/L_6, that is, M has the following structure



Observe that $\mathrm{LL}(M) = \mathrm{LL}(A) = 3$, and that M is not a rigid module. Consider the epic $\pi : P_1 \longrightarrow M$ and note that the simple module L_4 can be embedded in M. It is not difficult to check that the epic

$$\begin{bmatrix}\pi & 1_{L_4}\end{bmatrix} : P_1 \oplus L_4 \longrightarrow M \tag{3.7}$$

is a right minimal $\operatorname{add} G$-approximation of M. This map is not a projective cover of M. This shows that the claim of Corollary 3.3.1 does not usually hold for modules which are not rigid.

Using the approximation (3.7), one easily sees that R_A-module $\mathrm{Hom}_A(G, M)$ can be represented as

$$\begin{array}{cccc} & (1,3) & & (4,1) \\ \diagup & \mid & \diagdown & \mid \\ (2,1) & (3,2) & & (4,2) \\ & \mid & & \\ & (5,1) & & \end{array} \; .$$

Remark 3.3.4. Notice that Example 3.3.3 illustrates the statement of Proposition 3.3.2. The quiver presentation of ADR algebra studied in Section 2.7 also confirms Proposition 3.3.2 (note that the projective indecomposable modules over the algebra A in Section 2.7 are rigid).

3.3.2 An application of Theorem A

Motivated in part by the theory of quasihereditary algebras, Auslander, Platzeck and Todorov studied in [6] the homological properties of idempotent ideals. In this paper the authors defined a new class of algebras – the Artin algebras satisfying the descending Loewy length condition – and proved, in Theorem 7.3, [6], that every such algebra is quasihereditary.

Definition 3.3.5 ([6, Section 7])**.** An Artin algebra B satisfies the *descending Loewy length condition* (DLL condition, for short) if for every M in $\operatorname{mod} B$, a minimal projective resolution

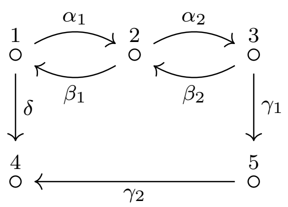
$$\cdots \longrightarrow P_i(M) \longrightarrow \cdots \longrightarrow P_0(M) \xrightarrow{\varepsilon} M \longrightarrow 0$$

satisfies $\mathrm{LL}(P_{i+1}(M)) < \mathrm{LL}(P_i(M))$, for all $i \geq 1$ such that $P_i(M) \neq 0$.

In [6] the authors claim that the Artin algebras of global dimension 2, the ADR algebras R_A, and the l-hereditary algebras (introduced in [49]) all satisfy the DLL condition. The main purpose of Theorem 7.3 in [6] was thus to give a unified proof of results in [20], [18] and [13], already established in the literature.

It is not difficult to check that Artin algebras of global dimension 2 and that l-hereditary algebras do satisfy the DLL condition. Unfortunately, it is not true that the ADR algebra R_A satisfies the DLL condition for every choice of A.

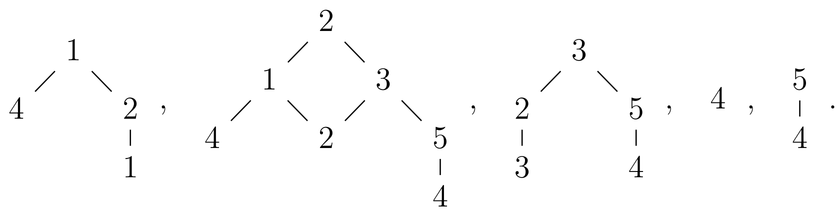
In order to see this, consider the following example: define $A := KQ/I$, where K is a field, Q is the quiver

$Q =$ 

and I is the admissible ideal

$$I = \langle \alpha_2\alpha_1,\ \beta_1\beta_2,\ \beta_2\alpha_2 - \alpha_1\beta_1,\ \delta\beta_1\alpha_1,\ \gamma_1\alpha_2\beta_2 \rangle.$$

Since I is generated by monomial relations and by commutative relations between paths of the same length, we may represent the projective indecomposable A-modules as



Note that the A-module L_3 is in the socle of P_3. Thus, using the labelling introduced in Section 2.2, the R_A-module $P_{3,3}$ contains a copy of $\Delta(3,1)$. The module $\Delta(3,1)$ has socle $L_{3,3}$, so we may consider the corresponding quotient module $M := P_{3,3}/L_{3,3}$.

Proposition 3.3.6. *Let A be the algebra introduced previously and consider the corresponding ADR algebra R_A. Let M be the R_A-module defined above. The DLL condition fails for the R_A-module M. Indeed, we have*

$$\mathrm{LL}(P_2(M)) \geq \mathrm{LL}(P_1(M)).$$

Proof. As $\mathrm{LL}(P_3) = 3$, we have that $\operatorname{Rad} P_{3,3}$ equals $\operatorname{Hom}_A(G, \operatorname{Rad} P_3)$ (see Proposition 2.3.1). Since $\operatorname{Rad} P_3$ is rigid, Corollary 3.3.1 implies that the minimal projective presentation of $L_{3,3}$ is of the form

$$P_{2,2} \oplus P_{5,2} \longrightarrow P_{3,3} \longrightarrow L_{3,3} \longrightarrow 0.$$

We claim that $\mathrm{LL}(P_{2,2}) \geq \mathrm{LL}(P_{3,3})$. Note that this will imply the statement in the proposition, as $P_{i+1}(M) = P_i(L_{3,3})$.

We start by showing that $\mathrm{LL}(P_{3,3}) = 5$. Observe that

$$\begin{aligned}\mathrm{LL}(P_{3,3}) &= 1 + \mathrm{LL}(\mathrm{Rad}\, P_{3,3}) \\ &= 1 + \mathrm{LL}(\mathrm{Hom}_A(G, \mathrm{Rad}\, P_3)) \\ &= 1 + \max\{\mathrm{LL}(\mathrm{Hom}_A(G, M_1)), \mathrm{LL}(\mathrm{Hom}_A(G, M_2))\},\end{aligned}$$

where $\mathrm{Rad}\, P_3 = M_1 \oplus M_2$,

$$M_1 := \begin{matrix} 2 \\ | \\ 3 \end{matrix}, \quad M_2 := \begin{matrix} 5 \\ | \\ 4 \end{matrix}.$$

By Lemmas 3.2.14 and 3.2.15, the chain of inclusions

$$0 \subset \mathrm{Hom}_A(G, L_4) \subset \mathrm{Hom}_A(G, M_2)$$

is the Δ-semisimple filtration of $\mathrm{Hom}_A(G, M_2)$, and it has factors $\Delta(4,1) = L_{4,1}$ and $\Delta(5,2) = L_{5,2}$, respectively. So $\mathrm{Hom}_A(G, M_2)$ has Loewy length 2. Consider now the module $N_1 := \mathrm{Hom}_A(G, M_1)$. We have that $\mathrm{LL}(N_1) = 4$ (we prove this in Lemma 3.3.8), therefore $\mathrm{LL}(P_{3,3}) = 5$. As we shall see in Lemma 3.3.9, $\mathrm{LL}(P_{2,2}) \geq 5$. This proves the result. □

For the proof of Lemmas 3.3.8 and 3.3.9, it will be useful to understand the extensions of the simple R_A-module $L_{2,4}$. This is addressed in the auxiliary remark below.

Remark 3.3.7. Consider the projective indecomposable R_A-module $P_{2,4}$. According to Lemmas 3.2.14 and 3.2.15, $P_{2,4}$ has a Δ-semisimple filtration

$$0 \subset \mathrm{Hom}_A(G, \mathrm{Soc}\, P_2) \subset \mathrm{Hom}_A(G, \mathrm{Soc}_2\, P_2) \subset \mathrm{Hom}_A(G, \mathrm{Soc}_3\, P_2) \subset P_{2,4}, \tag{3.8}$$

whose factors (ordered from the top to the bottom) are

$$\begin{gathered}\Delta(2,4) = L_{2,4}, \\ \Delta(3,3), \\ \Delta(1,2) \oplus \Delta(5,2), \\ \Delta(4,1) \oplus \Delta(2,1) \oplus \Delta(4,1).\end{gathered}$$

Using (3.8), it is not so easy to determine the precise elements $(k,l) \in \Lambda$ for which $\operatorname{Ext}^1_{R_A}(L_{2,4}, L_{k,l}) \neq 0$. However, by looking at (3.8), we conclude that

$$(k,l) \in \{(3,3),(1,2),(5,2),(4,1),(2,1)\}$$

in order for $\operatorname{Ext}^1_{R_A}(L_{2,4}, L_{k,l})$ to be nonzero.

Lemma 3.3.8. *Using the previous notation, let* $N_1 = \operatorname{Hom}_A(G, M_1)$ *be the* R_A*-module defined in the proof of Proposition 3.3.6. We have* $\operatorname{LL}(N_1) = 4$.

Proof. By Lemmas 3.2.14 and 3.2.15, the module N_1 has a Δ-semisimple filtration

$$0 \subset \operatorname{Hom}_A(G, L_3) \subset N_1,$$

with factors $\Delta(3,1)$ and $\Delta(2,2)$. In particular, N_1 has socle $L_{3,3}$. Consider now the pullback diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Delta(3,1) & \longrightarrow & N_2' & \longrightarrow & L_{2,4} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Delta(3,1) & \longrightarrow & N_1 & \longrightarrow & \Delta(2,2) & \longrightarrow & 0 \end{array} . \tag{3.9}$$

There is a (unique) submodule N_3 of N_1 with $\operatorname{Top} N_3 = L_{2,4}$. In fact, by looking at the diagram above we see that $N_3 \subseteq N_2'$, so

$$\operatorname{Rad} N_3 \subseteq \operatorname{Rad} N_2' \subseteq \Delta(3,1).$$

Using Remark 3.3.7 (and the structure of $\Delta(3,1)$), we conclude that $\operatorname{Rad} N_3$ must have top $L_{3,3}$. That is, the module N_3 has the following structure

$$N_3 = \begin{array}{c} (2,4) \\ | \\ (3,3) \end{array} .$$

In particular, the summand $L_{2,4}$ must appear in the socle of $N_1/\operatorname{Soc} N_1$. Now note that the exact sequence in the bottom of (3.9) induces the short exact sequence

$$0 \longrightarrow \Delta(3,1)/L_{3,3} \longrightarrow N_1/\operatorname{Soc} N_1 \longrightarrow \Delta(2,2) \longrightarrow 0,$$

so we must have

$$\operatorname{Soc}_2 N_1/\operatorname{Soc} N_1 = L_{3,2} \oplus L_{2,4}.$$

With this in mind, we may construct the first two rows of the commutative diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Soc}_2 \Delta(3,1) & \longrightarrow & \operatorname{Soc}_2 N_1 & \longrightarrow & L_{2,4} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Delta(3,1) & \longrightarrow & N_1 & \longrightarrow & \Delta(2,2) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L_{3,1} & \longrightarrow & N_1/\operatorname{Soc}_2 N_1 & \longrightarrow & \Delta(2,2)/L_{2,4} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
\quad . \tag{3.10}
$$

The exactness of the bottom row comes from the fact that the functor $1/\operatorname{Soc}_2(-)$ preserves monics (see Lemma 1.3.9). We want to prove that $\mathrm{LL}(N_1/\operatorname{Soc}_2 N_1) = 2$. Looking at the bottow row of (3.10) we see that the Loewy length of $N_1/\operatorname{Soc}_2 N_1$ is either 2 or 3. Note that $\mathrm{LL}(N_1/\operatorname{Soc}_2 N_1) = 3$ if and only if $\operatorname{Soc}_3 N_1/\operatorname{Soc}_2 N_1 = L_{3,1}$. If we had $\operatorname{Soc}_3 N_1/\operatorname{Soc}_2 N_1 = L_{3,1}$ then it would follow that $\operatorname{Ext}^1_{R_A}(L_{2,3}, L_{3,1}) \neq 0$, which cannot happen because $P_2/\operatorname{Rad}^3 P_2$ is rigid (see Proposition 3.3.2). Hence $\mathrm{LL}(N_1/\operatorname{Soc}_2 N_1) = 2$. So $\mathrm{LL}(N_1) = 4$. □

Lemma 3.3.9. *Using the previous notation, the projective R_A-module $P_{2,2}$ satisfies $\mathrm{LL}(P_{2,2}) \geq 5$.*

Proof. By Lemmas 3.2.14 and 3.2.15, the module $P_{2,2}$ has a Δ-semisimple filtration

$$0 \subset \Delta(1,1) \oplus \Delta(3,1) \subset P_{2,2},$$

with top factor $\Delta(2,2)$. The factor module $N_1' := P_{2,2}/\Delta(3,1)$ is indecomposable with socle $L_{1,3}$ (if it had socle $L_{1,3} \oplus L_{2,4}$ then, by Lemma 3.2.2, it would be a decomposable Δ-semisimple module). We assert that $N_1'/\operatorname{Soc} N_1'$ has simple socle $L_{1,2}$. Consider the pullback diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \Delta(1,1) & \longrightarrow & N_2' & \longrightarrow & L_{2,4} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Delta(1,1) & \longrightarrow & N_1' & \longrightarrow & \Delta(2,2) & \longrightarrow & 0
\end{array}
\quad .
$$

As before, there is a (unique) submodule N_3 of N_1' with $\operatorname{Top} N_3 = L_{2,4}$. Indeed, we must have $N_3 \subseteq N_2'$, and consequently

$$\operatorname{Rad} N_3 \subseteq \operatorname{Rad} N_2' \subseteq \Delta(1,1).$$

From Remark 3.3.7, we conclude that $\operatorname{Rad} N_3$ has simple top $L_{1,2}$. So N_3 must have the following structure

$$\begin{array}{c} (2,4) \\ | \\ (1,2) \\ | \\ (1,3) \end{array} .$$

Now it is not difficult to deduce that $\mathrm{LL}(N_1') \geq 5$. Hence $\mathrm{LL}(P_{2,2}) \geq 5$ (indeed this is an equality). □

By what we have just seen, the DLL condition does not hold for the ADR algebra R_A in general. However, the algebra R_A actually satisfies a property very similar to the DLL condition. The next results are implicitly proved in [5], within the proof of Proposition 10.2. We prove them for completeness. In the next result, the generator $G_{A/\operatorname{Rad}^m A}$ of $A/\operatorname{Rad}^m A$ is as defined in (2.1).

Lemma 3.3.10. *Let M be in* $\operatorname{mod} A$. *Suppose that* $\mathrm{LL}(M) = m$, *and let* $\varepsilon : X \longrightarrow M$ *be the right minimal* $\operatorname{add} G_{A/\operatorname{Rad}^m A}$*-approximation of M. Then ε is the right minimal* $\operatorname{add} G$*-approximation of M and* $\mathrm{LL}(\operatorname{Ker} \varepsilon) < m = \mathrm{LL}(X)$.

Proof. Let M be a module in $\operatorname{mod} A$ satisfying $\mathrm{LL}(M) = m$, and let

$$\varepsilon : X \longrightarrow M$$

be the right minimal $\operatorname{add} G_{A/\operatorname{Rad}^m A}$-approximation of M. Let $f : X' \longrightarrow M$ be a map such that $X' \in \operatorname{add} G$. Note that $\mathrm{LL}(\operatorname{Im} f) \leq m$, so the epic $X' \longrightarrow \operatorname{Im} f$ factors through the largest factor module of X' with Loewy length not greater than m. In other words, the map f must factor through $X'/\operatorname{Rad}^m X'$, and this module lies in $\operatorname{add} G_{A/\operatorname{Rad}^m A}$. Since ε is a right $\operatorname{add} G_{A/\operatorname{Rad}^m A}$-approximation, it follows that f factors through ε. Thus, the map ε is a right minimal $\operatorname{add} G$-approximation of M.

We now check that $\mathrm{LL}(\operatorname{Ker} \varepsilon) < m$. Note that $\mathrm{LL}(X) = m$ and $\mathrm{LL}(\operatorname{Ker} \varepsilon) \leq m$. Let $X = X' \oplus X''$ be a decomposition of X such that X'' is the direct sum of all indecomposable summands of X with Loewy length m. Let Y be an indecomposable summand of X'' and consider the map

$$\operatorname{Ker} \varepsilon \hookrightarrow X \stackrel{\pi}{\twoheadrightarrow} Y \quad , \tag{3.11}$$

where π is the projection map. The map (3.11) cannot be epic, otherwise it would split and ε would not be a right minimal morphism. Hence the image of the morphism (3.11) is contained in $\operatorname{Rad} Y$. Thus $\operatorname{Ker} \varepsilon \subseteq X' \oplus \operatorname{Rad} X''$. Therefore $\operatorname{LL}(\operatorname{Ker} \varepsilon) < m$. ☐

Proposition 3.3.11. *For every N in $\operatorname{mod} R_A$ there is an exact sequence of A-modules*

$$0 \longrightarrow X_t \longrightarrow \cdots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0 \ ,$$

with X_i in $\operatorname{add} G$ satisfying $\operatorname{LL}(X_{i+1}) < \operatorname{LL}(X_i)$ for all $i \geq 1$, such that

$$0 \longrightarrow \operatorname{Hom}_A(G, X_t) \longrightarrow \cdots \longrightarrow \operatorname{Hom}_A(G, X_0) \xrightarrow{\varepsilon} N \longrightarrow 0$$

is a minimal projective resolution for N.

Proof. Recall that the projective cover of an R_A-module $\operatorname{Hom}_A(G, M)$, $M \in \operatorname{mod} A$, is the image of the right minimal $\operatorname{add} G$-approximation of M' in $\operatorname{mod} A$ through the functor $\operatorname{Hom}_A(G, -)$.

Step 1 Using Lemma 3.3.10 inductively we conclude that there is an exact sequence in $\operatorname{mod} A$

$$0 \longrightarrow X_t \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow M \longrightarrow 0 \ ,$$

with X_i in $\operatorname{add} G$ satisfying $\operatorname{LL}(M) = \operatorname{LL}(X_0)$ and $\operatorname{LL}(X_{i+1}) < \operatorname{LL}(X_i)$ for all i, whose image through $\operatorname{Hom}_A(G, -)$ is the minimal projective resolution of $\operatorname{Hom}_A(G, M)$ in $\operatorname{mod} R_A$.

Step 2 Consider now a module N in $\operatorname{mod} R_A$, and let

$$P_1(N) \xrightarrow{p_{1*}} P_0(N) \longrightarrow N \longrightarrow 0.$$

be its minimal projective presentation. The map p_{1*} is the image of a map p_1 in $\operatorname{mod} A$ with domain (and codomain) in $\operatorname{add} G$. Suppose p_1 is nonzero with domain X. By the same reasoning as before we conclude that $\operatorname{LL}(\operatorname{Ker} p_1) < \operatorname{LL}(X)$ (note that the map p_1 must be right minimal). Now note that $\operatorname{Hom}_A(G, \operatorname{Ker} p_1) = \operatorname{Ker} p_{1*}$, so $P_i(\operatorname{Hom}_A(G, \operatorname{Ker} p_1)) = P_{i+2}(N)$. The statement of the proposition follows by applying the conclusions of Step 1 to the module $\operatorname{Ker} p_1$. ☐

Auslander proved in [5, Theorem 10.3] that the ADR algebra R_A has always finite global dimension not greater than $\operatorname{LL}(A)$. This result also follows from Proposition 3.3.11.

Corollary 3.3.12. *The algebra R_A is such that* $\operatorname{gl.dim} R_A \leq \operatorname{LL}(A)$.

3.4 Theorem B

The goal of this section is to prove Theorem B stated in the beginning of this chapter. In order to attain this, we investigate in detail the Δ-filtrations of the tilting modules over the ADR algebra R_A.

3.4.1 Motivation

Given an Artin algebra A, define

$$C = C_A := \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{\mathrm{LL}(Q_i)} \operatorname{Soc}_j Q_i.$$

This is a cogenerator of $\operatorname{mod} A$. Set $S_A := \operatorname{End}_A (C)^{op}$. It turns out that the algebras S_A and $\mathcal{R}(R_A)$ have a very similar structure. In fact, the statement of Theorem B can be loosely rephrased as: the algebras S_A and $\mathcal{R}(R_A)$ are isomorphic provided that A is "nice enough". Before delving into the technical results necessary to prove Theorem B, we will try to illustrate (informally) why the algebras S_A and $\mathcal{R}(R_A)$ should be related.

The algebra $\mathcal{R}(R_A)$ is quasihereditary with respect to $(\Lambda, \trianglelefteq^{op})$, where $(\Lambda, \trianglelefteq)$ is the poset associated with the algebra R_A described in (2.2) and (2.3). Recall the results in Section 2.6 concerning the Ringel dual of (right) ultra strongly quasihereditary algebras. According to these, $(\mathcal{R}(R_A), \Lambda, \trianglelefteq^{op})$ is a LUSQ algebra (see Theorem 2.6.1).

Turning the attention to the algebra S_A, we have that

$$\begin{aligned} S_A &= \operatorname{End}_A \left(\bigoplus_{i=1}^{n} \bigoplus_{j=1}^{\mathrm{LL}(Q_i)} \operatorname{Soc}_j Q_i \right)^{op} \\ &= \operatorname{End}_A \left(D \left(\bigoplus_{i=1}^{n} \bigoplus_{j=1}^{\mathrm{LL}(Q_i)} P_i^{A^{op}} / \operatorname{Rad}^j P_i^{A^{op}} \right) \right)^{op} \\ &\cong \operatorname{End}_{A^{op}} (G_{A^{op}}) = (R_{A^{op}})^{op}, \end{aligned}$$

where D is the standard duality and $P_i^{A^{op}}$ denotes the projective indecomposable A^{op}-module $D(Q_i)$. To avoid ambiguity, denote the poset corresponding to the ADR algebra $R_{A^{op}}$ of A^{op} by $(\Lambda_{A^{op}}, \trianglelefteq_{A^{op}})$ and represent its elements by $[i, j]$. Note that S_A is quasihereditary, as $R_{A^{op}}$ is. To be precise, $(S_A, \Lambda_{A^{op}}, \trianglelefteq_{A^{op}})$ is a LUSQ algebra since $(R_{A^{op}}, \Lambda_{A^{op}}, \trianglelefteq_{A^{op}})$ is RUSQ.

So both $(\mathcal{R}(R_A), \Lambda, \trianglelefteq^{op})$ and $(S_A, \Lambda_{A^{op}}, \trianglelefteq_{A^{op}})$ are LUSQ algebras. We take this analogy further by comparing the posets

$$(\Lambda, \trianglelefteq^{op}), \; \Lambda = \{(i,j) : 1 \leq i \leq n, \, 1 \leq j \leq l_i = \mathrm{LL}(P_i)\},$$
$$(\Lambda_{A^{op}}, \trianglelefteq_{A^{op}}), \; \Lambda_{A^{op}} = \{[i,j] : 1 \leq i \leq n, \, 1 \leq j \leq \mathrm{LL}(P_i^{A^{op}}) = \mathrm{LL}(Q_i)\}.$$

For the algebras $\mathcal{R}(R_A)$ and S_A to be isomorphic they must have the same number of simple modules, i.e. the sets Λ and $\Lambda_{A^{op}}$ must have the same cardinality. It seems then reasonable to require that $\mathrm{LL}(P_i) = \mathrm{LL}(Q_i)$, for all $1 \leq i \leq n$.

Ideally, an isomorphism between $\mathcal{R}(R_A)$ and S_A would somehow preserve the orders $\trianglelefteq^{op}$ and $\trianglelefteq_{A^{op}}$ of Λ and $\Lambda_{A^{op}}$, respectively. As $\mathcal{R}(R_A)$ and S_A are LUSQ algebras, they both have uniserial costandard modules. The costandard $\mathcal{R}(R_A)$-module with label (i,j), $\nabla'(i,j)$, has the following structure

$$\begin{matrix} (i,1) \\ | \\ (i,2) \\ | \\ \vdots \\ | \\ (i,j) \end{matrix}$$

(see Theorem 2.6.1). The costandard S_A-module with label $[i,j]$, $\nabla_{S_A}[i,j]$, is isomorphic to $D(\Delta_{A^{op}}[i,j])$, where $\Delta_{A^{op}}[i,j]$ is the standard $R_{A^{op}}$-module with label $[i,j]$. Thus the submodule lattice of $\nabla_{S_A}[i,j]$ is 'dual' to the submodule lattice of $\Delta_{A^{op}}[i,j]$. That is, $\nabla_{S_A}[i,j]$ has the following structure

$$\begin{matrix} [i, \mathrm{LL}(Q_i)] \\ | \\ [i, \mathrm{LL}(Q_i) - 1] \\ | \\ \vdots \\ | \\ [i,j] \end{matrix} .$$

If we suppose that $\mathrm{LL}(P_i) = \mathrm{LL}(Q_i) = l_i$ for all i, then the modules $\nabla'[i,j]$ and $\nabla_{S_A}(i, l_i - j + 1)$ have the same length for every $1 \leq i \leq n$, $1 \leq j \leq l_i$. The stronger assumption that $\mathrm{LL}(P_i) = \mathrm{LL}(Q_i) = L$ for all i, actually implies that the bijection $[i,j] \longmapsto (i, L - j + 1)$ preserves the partial orders. In this case, we have

$$\begin{aligned} [i,j] \lhd_{A^{op}} [k,l] &\Leftrightarrow j > l \\ &\Leftrightarrow L - j + 1 < L - l + 1 \\ &\Leftrightarrow (i, L - j + 1) \rhd (k, L - l + 1) \Leftrightarrow (i, L - j + 1) \lhd^{op} (k, L - l + 1). \end{aligned}$$

These observations support the assumptions and the claim of Theorem B.

Theorem B. *Suppose that A satisfies* $\mathrm{LL}(P_i) = \mathrm{LL}(Q_i) = L$ *for all i, $1 \leq 1 \leq n$. Moreover, suppose that all projectives P_i and all injectives Q_i are rigid. Then*

$$\mathcal{R}\left(R_A\right) \cong S_A \cong (R_{A^{op}})^{op}.$$

The rest of this chapter is mainly concerned with the proof of Theorem B. We refer to Appendix A for a complete example which illustrates the claim of Theorem B, and the preliminary results contained in Subsection 3.4.2.

3.4.2 Preliminary results

Roughly speaking, the quasihereditary structure of the algebra S_A depends on the socle series of the injective indecomposable A-modules, whereas the structure of the algebra $\mathcal{R}\left(R_A\right)$ depends on the filtrations

$$0 \subset T\left(i, l_i\right) \subset \cdots \subset T\left(i, j\right) \subset \cdots \subset T\left(i, 1\right) = Q_{i,l_i}.$$

The next results explore the connections between these two filtrations. These results will be needed for the proof of Theorem B.

Lemma 3.4.1. *Let A be such that* $\mathrm{LL}(P_i) = L$ *for all i, $1 \leq i \leq n$. Then*

$$T\left(k, L\right) = L_{k,L} = \Delta\left(k, L\right),$$

for $1 \leq k \leq n$.

Proof. The composition factor $L_{k,l}$ has multiplicity one in both $\Delta\left(k, l\right)$ and $T\left(k, l\right)$, and the composition factors of these two modules are of the form $L_{i,j}$, with $(i, j) \trianglelefteq (k, l)$. The lemma follows from the fact that (k, L) is a minimal element in $(\Lambda, \trianglelefteq)$. □

Proposition 3.4.2. *Let A be such that* $\mathrm{LL}(P_i) = L$ *for all i, $1 \leq i \leq n$. Then, for every (k, l) in Λ, we have*

$$T\left(k, l\right) \subseteq \mathrm{Hom}_A\left(G, \mathrm{Soc}_{L-l+1} Q_k\right) = \delta_{L-l+1}\left(Q_{k,L}\right) = \delta_{L-l+1}\left(T\left(k, 1\right)\right).$$

Proof. Recall that $T\left(k, 1\right) = Q_{k,L} = \mathrm{Hom}_A\left(G, Q_k\right)$ (see Lemma 2.4.4). Lemma 3.2.15 implies that

$$\mathrm{Hom}_A\left(G, \mathrm{Soc}_{L-l+1} Q_k\right) = \delta_{L-l+1}\left(Q_{k,L}\right) = \delta_{L-l+1}\left(T\left(k, 1\right)\right).$$

From Proposition 2.5.8, it follows that $T(k,l) \subseteq T(k,1)$. Hence, according to Lemma 1.3.21, it is enough to show that $\Delta.\operatorname{ssl} T(k,l) \leq L-l+1$ in order to establish the statement in the proposition (recall that $l^{(\delta,\bullet)}(-)$ corresponds to $\Delta.\operatorname{ssl}(-)$ in the present context).

Lemma 3.4.1 implies that $T(k,L) = \Delta(k,L)$. So $\Delta.\operatorname{ssl} T(k,L) = 1$. We prove that $\Delta.\operatorname{ssl} T(k,l) \leq L-l+1$ by descending induction on l.

Assume that $\Delta.\operatorname{ssl} T(x,l+1) \leq L-l$, for every x, and consider the short exact sequence

$$0 \longrightarrow \Delta(k,l) \xrightarrow{\phi} T(k,l) \longrightarrow X(k,l) \longrightarrow 0 \quad .$$

Recall that $X(k,l)$ lies in $\mathcal{F}(\Delta)$ and all its composition factors are of the form $L_{x,y}$ with $y \geq l+1$. Let

$$\bigoplus_{i\in I} Q_{x_i,L} := Q_0$$

be the injective hull of $X(k,l)$ (see part 5 of Proposition 2.5.6). Then $X(k,l)$ is contained in the largest submodule of Q_0 all of whose composition factors are of the form $L_{x,y}$, with $y \geq l+1$, i.e. we have

$$\begin{aligned} X(k,l) \subseteq \operatorname{Rej}\left(Q_0, \bigoplus_{(x,y):\, y<l+1} Q_{x,y}\right) &= \bigoplus_{i\in I} \operatorname{Rej}\left(Q_{x_i,L}, \bigoplus_{(x,y):\, (x,y)\rhd(x_i,l+1)} Q_{x,y}\right) \\ &= \bigoplus_{i\in I} T(x_i,l+1), \end{aligned}$$

where the last equality follows from Lemma 2.5.10. By induction and by part 1 of Lemma 1.3.21, we have that $\Delta.\operatorname{ssl} X(k,l) \leq L-l$. Note that $\Delta(k,l) = \delta(T(k,l))$. Lemma 1.3.19 implies that $\Delta.\operatorname{ssl} T(k,l) \leq L-l+1$. □

By Proposition 3.4.2, if all the projectives in $\operatorname{mod} A$ have the same Loewy length, then $T(k,l)$ is a submodule of $\delta_{L-l+1}(Q_{k,l})$ for every (k,l) in Λ. If additionally all projectives P_i are rigid, then a Δ-filtration of $T(k,l)$ has the same number of factors as a Δ-filtration of $\delta_{L-l+1}(Q_{k,L})$.

Proposition 3.4.3. *Suppose that A satisfies $\operatorname{LL}(P_i) = L$ for all i, $1 \leq i \leq n$. Assume that the projectives P_i are rigid. Then the monic*

$$\operatorname{Hom}_{R_A}(P_{i,L}, T(k,l)) \hookrightarrow \operatorname{Hom}_{R_A}(P_{i,L}, \delta_{L-l+1}(Q_{k,L}))$$

induced by the inclusion

$$T(k,l) \subseteq \delta_{L-l+1}(Q_{k,L}) = \delta_{L-l+1}(T(k,1)) = \operatorname{Hom}_A(G, \operatorname{Soc}_{L-l+1} Q_k)$$

is an isomorphism. In particular, the modules $T(k,l)$ and $\delta_{L-l+1}(Q_{k,L})$ are filtered by the same number of standard modules.

Proof. Recall the statement of Proposition 3.4.2 and consider an arbitrary map

$$f_* : P_{i,L} \longrightarrow \delta_{L-l+1}(Q_{k,L}).$$

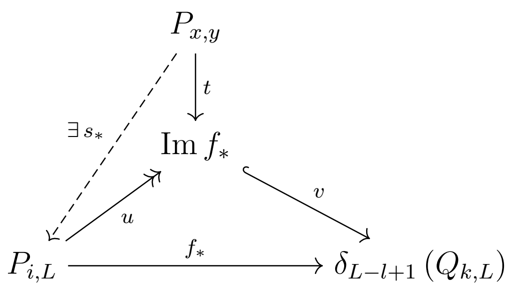
We claim that $\operatorname{Im} f_* \subseteq T(k,l)$ – note that once we prove this, the statement in the proposition follows. Our claim holds for $l = L$, as

$$\delta_1(Q_{k,L}) = \delta(T(k,1)) = \Delta(k,1),$$

and $T(k,L) = L_{k,L}$ (see Lemma 3.4.1). For the general case, by Lemma 2.5.10, we must show that all composition factors of $\operatorname{Im} f_*$ are of the form $L_{x,y}$, with $(x,y) \ntriangleright (k,l)$, that is, with $y \geq l$. So assume that $\operatorname{Im} f_* \neq 0$ and let

$$t : P_{x,y} \longrightarrow \operatorname{Im} f_*$$

be a nonzero map. We have the following commutative diagram



,

where the map s_* exists because $P_{x,y}$ is projective. We must have $f_* \circ s_* \neq 0$. As $\operatorname{Hom}_A(G,-)$ is a fully faithful functor, there is a nonzero map in $\operatorname{mod} A$

$$P_x/\operatorname{Rad}^y P_x \xrightarrow{\;s\;} P_i \xrightarrow{\;f\;} \operatorname{Soc}_{L-l+1} Q_k \quad ,$$

such that the functor $\operatorname{Hom}_A(G,-)$ takes f and s to f_* and s_*, respectively. Note that $\operatorname{LL}(\operatorname{Im} f) \leq L-l+1$, so $\operatorname{Rad}^{L-l+1} P_i \subseteq \operatorname{Ker} f$. Since P_i is rigid, then $\operatorname{Rad}^{L-l+1} P_i = \operatorname{Soc}_{l-1} P_i$. Now $\operatorname{Im} s$ is a submodule of P_i with Loewy length at most y, so $\operatorname{Im} s \subseteq \operatorname{Soc}_y P_i$. Since $f \circ s \neq 0$, we must have $y \geq l$. That is, $(k,l) \ntriangleleft (x,y)$. By a previous observation, $\operatorname{Im} f_*$ has to be contained in $T(k,l)$, and this finishes the proof. □

By combining Propositions 3.4.2 and 3.4.3 it is possible to compute the Δ-semisimple length of all tilting modules $T(k,l)$.

Lemma 3.4.4. *Suppose that A satisfies $\mathrm{LL}(P_i) = L$ for all i, $1 \leq i \leq n$, and assume that the projectives are rigid. Then*

$$\Delta.\operatorname{ssl} T\,(k,l) = \min\{L - l + 1, \mathrm{LL}(Q_k)\},$$

for $(k,l) \in \Lambda$. In particular, if $\mathrm{LL}(Q_i) = L$ for all i, then $\Delta.\operatorname{ssl} T\,(k,l) = L - l + 1$ for all $(k,l) \in \Lambda$.

Proof. For a module N in $\mathcal{F}(\Delta)$, denote the number of factors in a Δ-filtration of N by $|\Delta|(N)$. Set $\alpha = \min\{L - l + 1, \mathrm{LL}(Q_k)\}$.

By Proposition 3.4.2 and part 2 of Lemma 1.3.21 we have $\Delta.\operatorname{ssl} T\,(k,l) \leq L-l+1$. Since $T\,(k,l) \subseteq T\,(k,1) = \mathrm{Hom}_A\,(G, Q_k)$, then $\Delta.\operatorname{ssl} T\,(k,l) \leq \Delta.\operatorname{ssl}(\mathrm{Hom}_A\,(G, Q_k))$, by part 1 of Lemma 1.3.21. Lemma 3.2.15 implies that $\Delta.\operatorname{ssl}(\mathrm{Hom}_A\,(G, Q_k)) = \mathrm{LL}(Q_k)$. So $\Delta.\operatorname{ssl} T\,(k,l) \leq \alpha$.

Suppose, by contradiction, that $\Delta.\operatorname{ssl} T\,(k,l) \leq \alpha - 1$. Then

$$T\,(k,l) \subseteq \delta_{\alpha-1}\,(T\,(k,1)) = \delta_{\alpha-1}\,(Q_{k,L})\,,$$

by part 3 of Lemma 1.3.21. In particular, $|\Delta|(T\,(k,l)) \leq |\Delta|(\delta_{\alpha-1}\,(Q_{k,L}))$ (as $|\Delta|(N)$ coincides with the number of composition factors of $N \in \mathcal{F}(\Delta)$ of type $L_{i,L}$). By Lemma 3.2.15 we have

$$\delta_\alpha\,(Q_{k,L})\,/\delta_{\alpha-1}\,(Q_{k,L}) = \mathrm{Hom}_A\,(G, \operatorname{Soc}_\alpha Q_k)\,/\,\mathrm{Hom}_A\,(G, \operatorname{Soc}_{\alpha-1} Q_k)\,,$$

and this a nonzero Δ-semisimple module as $\alpha \leq \mathrm{LL}(Q_k) = \Delta.\operatorname{ssl} Q_{k,L}$. So

$$|\Delta|(T\,(k,l)) \leq |\Delta|(\delta_{\alpha-1}\,(Q_{k,L})) < |\Delta|(\delta_\alpha\,(Q_{k,L})) \leq |\Delta|(\delta_{L-l+1}\,(Q_{k,L})),$$

where the last inequality follows from the inclusion $\delta_\alpha\,(Q_{k,L}) \subseteq \delta_{L-l+1}\,(Q_{k,L})$. This contradicts Proposition 3.4.3. Hence $\Delta.\operatorname{ssl} T\,(k,l) = \alpha$. □

We now go one step further and fully describe the Δ-semisimple filtration of $T\,(k,l)$ in terms of the socle series of Q_k, in the case when all the projectives in $\operatorname{mod} A$ are rigid and have the same Loewy length.

Theorem 3.4.5. *Suppose that A satisfies $\mathrm{LL}(P_i) = L$ for all i, $1 \leq i \leq n$, and assume that the projectives are rigid. Let $(k,l) \in \Lambda$, and suppose that the socle layers of Q_k are*

$$\operatorname{Soc}_i Q_k / \operatorname{Soc}_{i-1} Q_k \cong \bigoplus_{\omega \in \Omega_i^k} L_{x_\omega},$$

for $i = 1, \ldots, \mathrm{LL}(Q_k)$. *Then*

$$\delta_i\left(T\left(k,l\right)\right)/\delta_{i-1}\left(T\left(k,l\right)\right) \cong \bigoplus_{\omega\in\Omega_i^k} \Delta\left(x_\omega, l+i-1\right),$$

for $i = 1, \ldots, \Delta.\,\mathrm{ssl}\, T\left(k,l\right)$.

Remark 3.4.6. As $\mathrm{Soc}\, Q_k = L_k$, then $|\Omega_1^k| = 1$ and $x_\omega = k$ for $\omega \in \Omega_1^k$.

Proof of Theorem 3.4.5. We prove the statement of the proposition by induction on i. Fix l, $1 \leq l \leq L$. Note that $\delta_1\left(T\left(k,l\right)\right) = \Delta\left(k,l\right)$, thus the claim holds trivially for $i = 1$ (see Remark 3.4.6). So let i be such that $2 \leq i \leq \Delta.\,\mathrm{ssl}\, T\left(k,l\right)$, and consider the short exact sequence

$$0 \longrightarrow Z_{i-1} \stackrel{\iota}{\longrightarrow} \delta_i\left(T\left(k,l\right)\right)/\delta_{i-2}\left(T\left(k,l\right)\right) \longrightarrow Z_i \longrightarrow 0,$$

where

$$Z_j := \delta_j\left(T\left(k,l\right)\right)/\delta_{j-1}\left(T\left(k,l\right)\right)$$

for $1 \leq j \leq \Delta.\,\mathrm{ssl}\, T\left(k,l\right)$. Suppose, by induction, that

$$Z_j = \bigoplus_{\omega\in\Omega_j^k} \Delta\left(x_\omega, l+j-1\right) \tag{3.12}$$

for $1 \leq j \leq i-1$. We wish to describe Z_i.

Step 1 We claim that the indecomposable summands of the Δ-semisimple module Z_i are all of the form $\Delta\left(x,y\right)$, with $y \geq l+i-1$. Suppose, by contradiction, that Z_i has a summand $\Delta\left(x,y\right)$, with $y \leq l+i-2$. Let $g : \Delta\left(x,y\right) \longrightarrow Z_i$ be a split monic, and let ν be such that $\nu \circ g = 1_{\Delta(x,y)}$. We then have the following pullback diagram

$$\begin{array}{ccccccccc}
0 \longrightarrow & \bigoplus_{\omega\in\Omega_{i-1}^k} \Delta\left(x_\omega, l+i-2\right) & \longrightarrow & N & \stackrel{f}{\longrightarrow} & \Delta\left(x,y\right) & \longrightarrow 0 \\
& \| & & \downarrow g' & & \downarrow g & \\
0 \longrightarrow & Z_{i-1} & \stackrel{\iota}{\longrightarrow} & \delta_i\left(T\left(k,l\right)\right)/\delta_{i-2}\left(T\left(k,l\right)\right) & \stackrel{\mathrm{coker}\,\iota}{\longrightarrow} & Z_i & \longrightarrow 0.
\end{array}$$

Note that $(x,y) \ntrianglelefteq (x_\omega, l+i-2)$ for every ω in Ω_{i-1}^k. By Lemma 1.4.5, there is a split monic μ such that $f \circ \mu = 1_{\Delta(x,y)}$. So

$$\nu \circ (\mathrm{coker}\,\iota) \circ g' \circ \mu = \nu \circ g \circ f \circ \mu = 1_{\Delta(x,y)},$$

that is, $\nu \circ (\mathrm{coker}\,\iota)$ is a split epic. This contradicts Lemma 3.2.11. Therefore, the summands of Z_i are all of the form $\Delta\left(x,y\right)$, with $y \geq l+i-1$.

Step 2 We may suppose that $l \geq 2$ as, by Lemmas 3.2.14 and 3.2.15, the result holds for $T(k,1) = \operatorname{Hom}_A(G, Q_k)$. Consider the family of commutative diagrams

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \delta_{j-1}(T(k,l)) & \longrightarrow & \delta_j(T(k,l)) & \longrightarrow & Z_j & \longrightarrow & 0 \\ & & \downarrow s_1^{(j)} & & \downarrow s_2^{(j)} & & \downarrow s_3^{(j)} & & \\ 0 & \longrightarrow & \delta_{j-1}(T(k,1)) & \longrightarrow & \delta_j(T(k,1)) & \longrightarrow & \bigoplus_{\omega \in \Omega_j^k} \Delta(x_\omega, j) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & W_{j-1} & \longrightarrow & W_j & \longrightarrow & \operatorname{Coker} s_3^{(j)} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}, \qquad (3.13)$$

for $j = 2, \ldots, i$. Here $s_1^{(j)} = \delta_{j-1}(s_2^{(j)})$, and the map $s_2^{(j)}$ is obtained by applying δ_j to the inclusion of $T(k,l)$ in $T(k,1)$. The morphism $s_3^{(j)}$ is a monic because the functor $1/\delta_{j-1}$ preserves monics (recall that δ_{j-1} is a hereditary preradical and use Lemma 1.3.9).

Let $j = i$ in (3.13). We claim that W_i has no composition factors of type $L_{x,L}$. As $i \leq \Delta.\operatorname{ssl} T(k,l)$, then $i \leq L - l + 1$ by Lemma 3.4.4. So $l \leq L - i + 1$. Thus

$$T(k, L-i+1) \subseteq T(k,l) \subseteq T(k,1).$$

Now, by Lemma 3.4.4, $\Delta.\operatorname{ssl} T(k, L-i+1) \leq i$, hence

$$T(k, L-i+1) \subseteq \delta_i(T(k,l)) \subseteq \delta_i(T(k,1)) = \operatorname{Hom}_A(G, \operatorname{Soc}_i Q_k).$$

Proposition 3.4.3 implies that $T(k, L-i+1)$ and $\operatorname{Hom}_A(G, \operatorname{Soc}_i Q_k)$ have the same number of composition factors of type $L_{x,L}$. By the chain of inclusions above, W_i and consequently $\operatorname{Coker} s_3^{(i)}$ have no composition factors of type $L_{x,L}$. So the Δ-semisimple module Z_i has the same socle as $\bigoplus_{\omega \in \Omega_j^k} \Delta(x_\omega, j)$. This fact, together with the conclusion of Step 1, implies that

$$Z_i = \bigoplus_{\omega \in \Omega_i^k} \Delta(x_\omega, y_\omega), \qquad (3.14)$$

with $L \geq y_\omega \geq l + i - 1$.

Step 3 Recall that i is such that $2 \leq i \leq \Delta.\operatorname{ssl} T(k,l) \leq L - l + 1$ (see Lemma 3.4.4 for the last inequality).

We wish to prove that $y_\omega = l + i - 1$ for every $\omega \in \Omega_i^k$ in the identity (3.14). Note that the identity must hold in the case $i = L - l + 1$. So we may assume that $i \leq L - l$, i.e. that $l + i \leq L$.

Suppose, by contradiction, that $y_{\omega'} \geq l + i$ for some $\omega' \in \Omega_i^k$. Consider again the diagram (3.13) for $j = i$. There is some element ω' in Ω_i^k satisfying $y_{\omega'} \geq l + i$, and such that the composition factor $L_{x_{\omega'}, y_{\omega'}-1}$ appears in the socle of $\operatorname{Coker} s_3^{(i)}$. The simple module $L_{x_{\omega'}, y_{\omega'}-1}$ is then a composition factor of W_i as well.

We have that $s_2^{(i)} = \delta_i(\kappa)$, where κ is the inclusion of $T(k,l)$ on $T(k,1)$. The functor δ_i is left exact by Lemma 1.3.9, so

$$W_i \subseteq \delta_i\left(T(k,1)/T(k,l)\right) = \delta_i\left(Q_{k,l-1}\right) \subseteq Q_{k,l-1},$$

and $W_i \neq 0$ (because $W_{i-1} \neq 0$ by induction). Since $y_{\omega'} > l+i-1 \geq l$, the composition factor $L_{x_{\omega'}, y_{\omega'}-1}$ cannot be in the socle of W_i. Thus, some composition factor of

$$\operatorname{Rad} P_{x_{\omega'}, y_{\omega'}-1} / \operatorname{Rad}^2 P_{x_{\omega'}, y_{\omega'}-1},$$

say $L_{\alpha,\beta}$, appears in the composition series of W_i. In fact, as $L_{x_{\omega'}, y_{\omega'}-1}$ is in the socle of $\operatorname{Coker} s_3^{(i)}$, then $L_{\alpha,\beta}$ has to be a composition factor of W_{i-1}. Proposition 3.3.2 implies that either $(\alpha,\beta) = (x_{\omega'}, y_{\omega'})$ or $(\alpha,\beta) = (x, y_{\omega'} - 2)$, for some x. In both cases, we have

$$\beta \geq y_{\omega'} - 2 \geq l + i - 2.$$

Step 4 Consider now the commutative diagrams (3.13) for $j = 2, \ldots, i-1$. By induction, one knows how the modules Z_j look like for $2 \leq j \leq i-1$ (see (3.12)). Thus, for $2 \leq j \leq i-1$, the composition factors of $\operatorname{Coker} s_3^{(j)}$ are $L_{x_\omega, j}, \ldots, L_{x_\omega, l+j-2}$, $\omega \in \Omega_j^k$. Moreover, $W_1 = \Delta(k,1)/\Delta(k,l)$. Hence all composition factors of W_{i-1} are of the form $L_{x,y}$, with $y \leq l + i - 3$. But then $L_{\alpha,\beta}$ cannot be a composition factor of W_{i-1} – a contradiction. Therefore, the assumption made in the beginning of Step 3 is wrong and we must have $y_\omega = l + i - 1$ for every ω in Ω_i^k. □

3.4.3 Proof of Theorem B

We finally prove the main result of this chapter.

Theorem B. *Suppose that A satisfies* $\mathrm{LL}(P_i) = \mathrm{LL}(Q_i) = L$ *for all i, $1 \leq i \leq n$. Moreover, suppose that all projectives P_i and all injectives Q_i are rigid. Then*

$$\mathcal{R}(R_A) \cong S_A \cong (R_{A^{op}})^{op}.$$

The proof presented in here follows a suggestion by K. Erdmann, and replaces our previous (much less elegant) method to establish this result. The two key ingredients for the proof of Theorem B (Propositions 3.4.7 and 3.4.9) rely on the description of the Δ-semisimple filtration of the tilting modules given in Theorem 3.4.5.

Recall that the underlying algebra A is an Artin C-algebra. Therefore the ADR algebra R_A is also an Artin C-algebra, and $\mathrm{Hom}_{R_A}(X, Y)$ lies in $\mathrm{mod}\, C$ for X and Y in $\mathrm{mod}\, R_A$.

Proposition 3.4.7. *Suppose that A satisfies* $\mathrm{LL}(P_i) = \mathrm{LL}(Q_i) = L$ *for all i, $1 \leq i \leq n$, and assume that all projectives P_i and all injectives Q_i are rigid. Then the C-modules* $\mathrm{Hom}_{R_A}(T(k,l), T(i,j))$ *and* $\mathrm{Hom}_{R_A}(\delta_{L-l+1}(Q_{k,L}), \delta_{L-j+1}(Q_{i,L}))$ *have the same (Jordan–Hölder) length.*

Proof. Denote the length of a module M in $\mathrm{mod}\, C$ by $l(M)$. We will use the notation in the statement of Theorem 3.4.5 to describe the socle layers of the injective indecomposable module Q_k.

We start by determining the value of $l(\mathrm{Hom}_{R_A}(T(k,l), T(i,j)))$. Using that the functor $\mathrm{Hom}_{R_A}(-, T(i,j))$ preserves exact sequences in $\mathcal{F}(\Delta)$ (see Remark 1.4.16) together with Theorem 3.4.5, we deduce that

$$l\left(\mathrm{Hom}_{R_A}\left(T\left(k,l\right), T\left(i,j\right)\right)\right) = \sum_{y=1}^{\Delta.\mathrm{ssl}\, T(k,l)} \sum_{\omega \in \Omega_y^k} l\left(\mathrm{Hom}_{R_A}\left(\Delta\left(x_\omega, l+y-1\right), T\left(i,j\right)\right)\right)$$

Lemma 3.4.4 implies that $\Delta.\,\mathrm{ssl}\, T(k,l) = L - l + 1$. By Theorem 2.5.8, the module $T(i,j)$ is filtered by the costandard modules $\nabla(i,j), \nabla(i,j+1), \ldots, \nabla(i,L)$. Using again that $\mathrm{Ext}^1_{R_A}(\mathcal{F}(\Delta), \mathcal{F}(\nabla)) = 0$ (see Lemma 1.4.8), we get

$$\begin{aligned}\sum_{y=1}^{L-l+1} \sum_{\omega \in \Omega_y^k} & l\left(\mathrm{Hom}_{R_A}\left(\Delta\left(x_\omega, l+y-1\right), T\left(i,j\right)\right)\right) \\ &= \sum_{y=1}^{L-l+1} \sum_{\omega \in \Omega_y^k} \sum_{z=j}^{L} l\left(\mathrm{Hom}_{R_A}\left(\Delta\left(x_\omega, l+y-1\right), \nabla\left(i,z\right)\right)\right).\end{aligned}$$

Note that

$$\sum_{y=1}^{L-l+1}\sum_{\omega\in\Omega_y^k}\sum_{z=j}^{L} l\left(\operatorname{Hom}_{R_A}\left(\Delta\left(x_\omega, l+y-1\right), \nabla\left(i,z\right)\right)\right)$$

$$= \sum_{y=1}^{L-l+1}\sum_{\omega\in\Omega_y^k}\sum_{z=j}^{L}\delta_{(x_\omega,l+y-1),(i,z)} l\left(\operatorname{End}_{R_A}\left(\Delta\left(x_\omega, l+y-1\right)\right)\right)$$

$$= \sum_{y=1}^{L-l+1}\sum_{\omega\in\Omega_y^k}\sum_{z=j}^{L}\delta_{(x_\omega,l+y-1),(i,z)} l\left(\operatorname{End}_{A}\left(L_{x_\omega}\right)\right)$$

$$= \sum_{y=\max\{j-l,0\}+1}^{L-l+1}\sum_{\omega\in\Omega_y^k}\delta_{x_\omega,i} l\left(\operatorname{End}_{A}\left(L_{x_\omega}\right)\right).$$

In here the first equality follows from Lemma 1.4.11, the second equality will follow from Lemma 3.4.8, and the third equality follows by analysing the values taken by the Kronecker delta.

Now we calculate $l(\operatorname{Hom}_{R_A}(\delta_{L-l+1}(Q_{k,L}), \delta_{L-j+1}(Q_{i,L})))$. Observe that

$$\operatorname{Hom}_{R_A}\left(\delta_{L-l+1}\left(Q_{k,L}\right), \delta_{L-j+1}\left(Q_{i,L}\right)\right)$$

$$= \operatorname{Hom}_{R_A}\left(\operatorname{Hom}_A\left(G, \operatorname{Soc}_{L-l+1} Q_k\right), \operatorname{Hom}_A\left(G, \operatorname{Soc}_{L-j+1} Q_i\right)\right)$$

$$\cong \operatorname{Hom}_A\left(\operatorname{Soc}_{L-l+1} Q_k, \operatorname{Soc}_{L-j+1} Q_i\right),$$

where the first equality follows from Proposition 3.4.2 and the second identity is due to the fact that $\operatorname{Hom}_A(G,-)$ is a fully faithful functor. Any map $f : \operatorname{Soc}_{L-l+1} Q_k \longrightarrow \operatorname{Soc}_{L-j+1} Q_i$ is such that $\operatorname{LL}(\operatorname{Im} f) \leq L-j+1$, so f must factor through the largest quotient of $\operatorname{Soc}_{L-l+1} Q_k$ whose Loewy length is at most $L-j+1$. That is, f factors through the module

$$\operatorname{Soc}_{L-l+1} Q_k / \operatorname{Rad}^{L-j+1}\left(\operatorname{Soc}_{L-l+1} Q_k\right) = \operatorname{Soc}_{L-l+1} Q_k / \operatorname{Soc}_{\max\{L-l+1-(L-j+1),0\}} Q_k,$$

where the equality follows from the rigidity of Q_k. So the canonical epic

$$\operatorname{Soc}_{L-l+1} Q_k \longrightarrow \operatorname{Soc}_{L-l+1} Q_k / \operatorname{Soc}_{\max\{j-l,0\}} Q_k$$

induces an isomorphism of C-modules

$$\operatorname{Hom}_A\left(\operatorname{Soc}_{L-l+1} Q_k / \operatorname{Soc}_{\max\{j-l,0\}} Q_k, \operatorname{Soc}_{L-j+1} Q_i\right)$$

$$\cong \operatorname{Hom}_A\left(\operatorname{Soc}_{L-l+1} Q_k, \operatorname{Soc}_{L-j+1} Q_i\right).$$

Notice that both $\operatorname{Soc}_{L-l+1} Q_k / \operatorname{Soc}_{\max\{j-l,0\}} Q_k$ and $\operatorname{Soc}_{L-j+1} Q_i$ are modules over $A/\operatorname{Rad}^{L-j+1} A$. In fact, $\operatorname{Soc}_{L-j+1} Q_i$ is an injective in $\operatorname{mod}(A/\operatorname{Rad}^{L-j+1} A)$. Thus the restriction of $\operatorname{Hom}_A(-, \operatorname{Soc}_{L-j+1} Q_i)$ to $\operatorname{mod}(A/\operatorname{Rad}^{L-j+1} A)$ yields an exact functor. Therefore

$$\begin{aligned} l\left(\operatorname{Hom}_{R_A}\left(\delta_{L-l+1}\left(Q_{k,L}\right), \delta_{L-j+1}\left(Q_{i,L}\right)\right)\right) \\ &= l\left(\operatorname{Hom}_A\left(\operatorname{Soc}_{L-l+1} Q_k/\operatorname{Soc}_{\max\{j-l,0\}} Q_k, \operatorname{Soc}_{L-j+1} Q_i\right)\right) \\ &= \sum_{y=\max\{j-l,0\}+1}^{L-l+1} \sum_{\omega\in\Omega_y^k} l\left(\operatorname{Hom}_A\left(L_{x_\omega}, \operatorname{Soc}_{L-j+1} Q_i\right)\right) \\ &= \sum_{y=\max\{j-l,0\}+1}^{L-l+1} \sum_{\omega\in\Omega_y^k} l\left(\operatorname{Hom}_A\left(L_{x_\omega}, L_i\right)\right) \\ &= \sum_{y=\max\{j-l,0\}+1}^{L-l+1} \sum_{\omega\in\Omega_y^k} \delta_{x_\omega,i} l\left(\operatorname{End}_A\left(L_{x_\omega}\right)\right), \end{aligned}$$

which shows that $\operatorname{Hom}_{R_A}(\delta_{L-l+1}(Q_{k,L}), \delta_{L-j+1}(Q_{i,L}))$ and $\operatorname{Hom}_{R_A}(T(k,l), T(i,j))$ have the same length over C. □

Lemma 3.4.8. *Let $(B, \Phi, \sqsubseteq)$ be a RUSQ algebra (over C). Then*

$$\operatorname{End}_B\left(\Delta\left(i, j+1\right)\right) \cong \operatorname{Hom}_B\left(\Delta\left(i, j+1\right), \Delta\left(i, j\right)\right) \cong \operatorname{End}_B\left(\Delta\left(i, j\right)\right)$$

as C-modules. If B is the ADR algebra R_A of an Artin algebra A then the modules above are isomorphic to $\operatorname{End}_A(L_i)$.

Proof. Consider the short exact sequence in $\operatorname{mod} B$

$$0 \longrightarrow \Delta\left(i, j+1\right) \longrightarrow \Delta\left(i, j\right) \longrightarrow L_{i,j} \longrightarrow 0.$$

By applying the functor $\operatorname{Hom}_B(\Delta(i, j+1), -)$ to this exact sequence we deduce that $\operatorname{End}_B(\Delta(i, j+1)) \cong \operatorname{Hom}_B(\Delta(i, j+1), \Delta(i, j))$. Using $\operatorname{Hom}_B(-, \Delta(i, j))$, we get an exact sequence

$$0 \longrightarrow \operatorname{End}_B\left(\Delta\left(i, j\right)\right) \longrightarrow \operatorname{Hom}_B\left(\Delta\left(i, j+1\right), \Delta\left(i, j\right)\right) \longrightarrow \operatorname{Ext}_B^1\left(L_{i,j}, \Delta\left(i, j\right)\right).$$

Note that $\operatorname{Ext}_B^1(L_{i,j}, \Delta(i,j)) = 0$. If this was not the case, there would exist a module M with socle L_{i,l_i}, having a unique composition factor of type L_{x,l_x} and satisfying $[M : L_{i,j}] = 2$. According to parts 4 and 5 of Proposition 2.5.6, M would have to be a standard module. This cannot happen as $[M : L_{i,j}] = 2$. This shows that the C-modules $\operatorname{End}_B(\Delta(i,j))$ and $\operatorname{Hom}_B(\Delta(i, j+1), \Delta(i,j))$ are isomorphic.

For the claim about R_A, recall that $\Delta(i, 1) = \operatorname{Hom}_A(G, L_i)$ (see Proposition 2.3.4). Since the functor $\operatorname{Hom}_A(G, -)$ is fully faithful, then $\operatorname{End}_{R_A}(\Delta(i,1)) \cong \operatorname{End}_A(L_i)$. □

By Proposition 3.4.2, the R_A-module $T(k,l)$ is contained in $\delta_{L-l+1}(Q_{k,L})$. We will show that the maps in $\operatorname{Hom}_{R_A}(\delta_{L-l+1}(Q_{k,L}), \delta_{L-j+1}(Q_{i,L}))$ give rise to maps in $\operatorname{Hom}_{R_A}(T(k,l), T(i,j))$ via restriction. This is the final piece needed to prove Theorem B.

Proposition 3.4.9. *Suppose that A satisfies $\mathrm{LL}(P_i) = \mathrm{LL}(Q_i) = L$ for all i, $1 \leq i \leq n$, and assume that all projectives P_i and all injectives Q_i are rigid. Consider a morphism*

$$f_* : \delta_{L-l+1}(Q_{k,L}) \longrightarrow \delta_{L-j+1}(Q_{i,L}).$$

Then $f_*(T(k,l)) \subseteq T(i,j)$.

Proof. Because $\operatorname{Hom}_A(G,-)$ is a full functor, then $f_* = \operatorname{Hom}_A(G,f)$ for a map $f : \operatorname{Soc}_{L-l+1} Q_k \longrightarrow \operatorname{Soc}_{L-j+1} Q_i$ in $\operatorname{mod} A$. Note that

$$\operatorname{Ker} f \supseteq \operatorname{Rad}^{L-j+1}(\operatorname{Soc}_{L-l+1} Q_k) = \operatorname{Soc}_{\max\{j-l,0\}} Q_k.$$

Since $T(k,l) \subseteq \delta_{L-l+1}(Q_{k,L})$, then

$$\delta_{\max\{j-l,0\}}(T(k,l)) \subseteq \delta_{\max\{j-l,0\}}(\delta_{L-l+1}(Q_{k,L})) = \delta_{\max\{j-l,0\}}(Q_{k,L}).$$

Observe that,

$$\delta_{\max\{j-l,0\}}(Q_{k,L}) = \operatorname{Hom}_A\left(G, \operatorname{Soc}_{\max\{j-l,0\}} Q_k\right) \subseteq \operatorname{Hom}_A(G, \operatorname{Ker} f) = \operatorname{Ker} f_*,$$

so $\delta_{\max\{j-l,0\}}(T(k,l))$ is contained in the kernel of $f_*|_{T(k,l)}$. In other words, $f_*(T(k,l))$ is isomorphic to a quotient of $N = T(k,l)/\delta_{\max\{j-l,0\}}(T(k,l))$. Theorem 3.4.5 implies that all composition factors of N are of the form $L_{x,y}$, with $y \geq l + \max\{j-l,0\} \geq j$. Therefore all composition factors of $f_*(T(k,l))$ are of the form $L_{x,y}$, with $(x,y) \not> (i,j)$. By Lemma 2.5.10, the module $f_*(T(k,l))$ must be contained in $T(i,j)$. □

Proof of Theorem B. Consider the morphism of Artin C-algebras

$$\varphi : \operatorname{End}_{R_A}\left(\bigoplus_{(i,j)\in\Lambda} \delta_j(Q_{i,L})\right) \longrightarrow \operatorname{End}_{R_A}\left(\bigoplus_{(i,j)\in\Lambda} T(i,j)\right) = \mathcal{R}(R_A)^{op},$$

which sends each map $g \in \operatorname{End}_{R_A}(\bigoplus_{i=1}^n \bigoplus_{j=1}^L \delta_{L-j+1}(Q_{i,L}))$ to the corresponding restriction to $\bigoplus_{i=1}^n \bigoplus_{j=1}^L T(i,j)$. According to Proposition 3.4.9, φ is well defined. Moreover, if $g \neq 0$ then $\varphi(g) \neq 0$, as the modules $\delta_{L-j+1}(Q_{i,L})$ have simple socle. So φ is an injective morphism of C-algebras, and in particular, a monomorphism of modules in $\operatorname{mod} C$. Proposition 3.4.7 implies that φ is a bijection.

As $\delta_j(Q_{i,L}) = \mathrm{Hom}_A(G, \mathrm{Soc}_j\, Q_i)$, then

$$\mathrm{End}_{R_A}\left(\bigoplus_{(i,j)\in\Lambda} \delta_j\left(Q_{i,L}\right)\right) \cong \mathrm{End}_A\left(\bigoplus_{i=1}^{n}\bigoplus_{j=1}^{L} \mathrm{Soc}_j\, Q_i\right) = (S_A)^{op},$$

using that $\mathrm{Hom}_A\left(G,-\right)$ is a fully faithful functor. Thus the algebras $\mathcal{R}(R_A)$ and S_A are isomorphic. The identity $S_A \cong (R_{A^{op}})^{op}$ was established in Subsection 3.4.1. □

Remark 3.4.10. Let A be an Artin algebra satisfying $A \cong A^{op}$. Suppose further that A has rigid projectives and injectives, and assume they all have the same Loewy length. Then, by Theorem B,

$$\mathcal{R}\left(R_A\right) \cong S_A \cong (R_{A^{op}})^{op} \cong (R_A)^{op}. \tag{3.15}$$

In particular, the identity (3.15) holds when A is the Brauer tree algebra studied in Section 2.7. Recall that, for a field K of prime characteristic, any nonsimple block of $K\Sigma_m$ of finite type is Morita equivalent to one of these Brauer tree algebras.

The principal block A of the group algebra $K\Sigma_{2p}$, where K is a field of odd characteristic p, is yet another example of a self-injective algebra with rigid projective-injectives having fixed Loewy length (see [48]). So the identity (3.15) also holds in this case.

Finally, note that (3.15) is also satisfied when A is a preprojective algebra of type A_n (see Subsection 5.3.3), as these are also self-injective algebras with rigid projective-injectives having constant Loewy length.

Chapter 4

Further observations

4.1 Overview of the chapter

In this rather disconnected chapter we answer further natural questions about the ADR algebra, and investigate a slightly weaker version of the class of (left) ultra strongly quasihereditary algebras. This chapter marks a transition of focus. Here we wrap up our remaining questions about the ADR algebra and lay the basis for some of the problems investigated in Chapter 5.

The first half of this chapter (Section 4.2) attempts to answer the following questions concerning the ADR algebra:

- To what extent is the ADR algebra R_A a "Schur algebra" for A?
- What is the representation type of the subcategories $\mathcal{F}(\Delta)$ and $\mathcal{F}(\nabla)$ of the category $\operatorname{mod} R_A$? Is this related to the representation type of A?
- When is the ADR algebra Ringel self-dual?
- Does the ADR algebra R_A provide any information about the finitistic dimension of A?

In the second half of this chapter (Section 4.3), we derive the main properties of a class of quasihereditary algebras which contains every LUSQ algebra (and in particular it includes the Ringel dual $\mathcal{R}(R_A)$ and the opposite algebra $(R_A)^{op}$ of the ADR algebra R_A of A). To be precise, we study the quasihereditary algebras $(B, \Phi, \sqsubseteq)$ such that the elements in Φ can be labelled as

$$\Phi = \{(i, j) : 1 \leq i \leq n,\, 1 \leq j \leq l_i\},$$

for some $n, l_i \in \mathbb{Z}_{>0}$, where

$$0 \subset P_{i,l_i} \subset \ldots \subset P_{i,2} \subset P_{i,1}$$

is the unique Δ-filtration of $P_{i,1}$. The results deduced in Section 4.3 will then be used in Chapter 5 to provide examples of (left) strongly quasihereditary algebras.

4.2 Further questions about the ADR algebra

We give partial answers to the previously stated questions.

4.2.1 The ADR algebra is not a 1-faithful quasihereditary cover

The general theory of n-faithful quasihereditary covers of an algebra was developed by Rouquier in [53] (see also [3, Section 4] for a self-contained approach to Rouquier's theory), in order to establish important results about rational Cherednik algebras. The highlight in [53] is the following theorem: the category $\mathcal{O}$ for a rational Cherednik algebra of type A is equivalent to the module category of a q-Schur algebra (for certain parameters q) when the base field is $\mathbb{C}$. The main idea behind the proof of this result is the uniqueness, up to Morita equivalence, of certain types of "quasihereditary covers" called 1-faithful quasihereditary covers.

Definition 4.2.1 ([53], [3, Definition 4.45])**.** Let A and B be finite-dimensional K-algebras[1] and let P be a projective in $\operatorname{mod} B$. Suppose that:

1. $A \cong \operatorname{End}_B(P)^{op}$;
2. B is quasihereditary with respect to some poset $(\Phi, \sqsubseteq)$ and $\operatorname{End}_B(\Delta(i)) \cong K$ for all $i \in \Phi$.

Then B is a *n-faithful quasihereditary cover* of A if the functor

$$F := \operatorname{Hom}_B(P, -) : \operatorname{mod} B \longrightarrow \operatorname{mod} A,$$

satisfies the condition

$$\operatorname{Ext}^i_B(X, Y) \cong \operatorname{Ext}^i_A(F(X), F(Y)).$$

for all X and Y in $\mathcal{F}(\Delta)$ and all i, $0 \leq i \leq n$.

The A-modules $S(i) := F(\Delta(i))$, $i \in \Phi$, are called the *Specht modules*.

[1] In [53], the underlying algebras in the definition of n-faithful quasihereditary cover are more general. We use finite-dimensional K-algebras for simplicity.

Loosely speaking, a pair (B, P) is a 1-faithful quasihereditary cover of $A \cong \mathrm{End}_B(P)^{op}$ if the Specht modules over A can be glued together in way which mimics the gluing of the standard modules over B. The definition of 1-faithful quasihereditary cover emulates classic results relating the Specht modules over the algebra of the symmetric group and the corresponding standard modules over the Schur algebra. A quasihereditary cover is n-faithful for higher values of n if the correspondence between the standard B-modules $\Delta(i)$ and the Specht A-modules $S(i)$ not only preserves the gluing but also the "relations" up to length n.

The importance of 1-faithful quasihereditary covers stems from the following result, due to Rouquier ([53]).

Theorem 4.2.2 ([53], [3, Theorem 4.50])**.** *Let $(B, \Phi, \sqsubseteq)$ and $(B', \Phi', \sqsubseteq')$ be quasi-hereditary algebras, and suppose that (B, P) and (B', P') are 1-faithful quasihereditary covers of A. If*

$$\{S(i) : i \in \Phi\} = \{S(i') : i' \in \Phi'\},$$

then $\mathrm{mod}\, B$ *and* $\mathrm{mod}\, B'$ *are equivalent categories.*

Consider now a finite-dimensional K-algebra A, and suppose that $\mathrm{End}_A(L_i) \cong K$ for every simple A-module L_i. It is not difficult to check that $\mathrm{End}_{R_A}(L_{i,j}) \cong K$ for every simple R_A-module $L_{i,j}$. Define $P := \mathrm{Hom}_A(G, A)$. Note that (R_A, P) is a 0-faithful quasihereditary cover of A – this follows from the fact that $\mathrm{Hom}_A(G, -)$ is right adjoint to the functor $\mathrm{Hom}_{R_A}(P, -)$. However, (R_A, P) is not usually a 1-faithful quasihereditary cover of A.

Proposition 4.2.3. *Using the previous notation, the pair (R_A, P) is a 1-faithful quasihereditary cover of A if and only if A is semisimple.*

Proof. If A is semisimple then the algebra R_A is Morita equivalent to A. So in this case R_A is trivially a 1-faithful quasihereditary cover of A.

If A is not semisimple, then, using the notation introduced in Section 2.2, there are simple A-modules L_i and L_k such that $\mathrm{Ext}^1_A(L_i, L_k) \neq 0$. Lemma 1.4.5 implies that $\mathrm{Ext}^1_{R_A}(\Delta(i,1), \Delta(k,1)) = 0$. Now recall that the module $\Delta(x,1)$ is isomorphic to $P_{x,1} = \mathrm{Hom}_A(G, L_x)$ (see Proposition 2.3.4), so

$$S(x,1) = F(\Delta(x,1)) = \mathrm{Hom}_{R_A}(P, \Delta(x,1)) \cong L_x.$$

Since $\mathrm{Ext}^1_A(F(\Delta(i,1)), F(\Delta(k,1))) \neq 0$, then (R_A, P) cannot be a 1-faithful quasi-hereditary cover of A. □

4.2.2 The representation type of $\mathcal{F}(\Delta)$ and $\mathcal{F}(\nabla)$

Recall that an Artin algebra A has *finite representation type* if there are only finitely many isomorphism classes of indecomposable modules in $\operatorname{mod} A$. In a similar way, we say that a subcategory $\mathcal{C}$ of $\operatorname{mod} A$ has *finite type* if there are only finitely many isomorphism classes of indecomposables in $\mathcal{C}$.

We would like to understand the relationship between the representation type of A and the representation type of R_A, $\mathcal{F}(\Delta)$ and $\mathcal{F}(\nabla)$. It is easy to conclude that A has finite representation type whenever R_A has finite type. However, the converse of this assertion is not true.

According to Lemma 2.3.6, the category $\operatorname{mod} A$ can be embedded in $\mathcal{F}(\Delta)$, and these categories seem to be quite similar. It would not be outlandish to expect $\operatorname{mod} A$ and $\mathcal{F}(\Delta)$ to have the same representation type.

In contrast with $\mathcal{F}(\Delta)$, the subcategory $\mathcal{F}(\nabla)$ of $\operatorname{mod} R_A$ seems to be quite small: Proposition 2.5.11 implies that the injective indecomposable R_A-modules are filtered by "very few" costandard modules.

It turns out that the categories $\mathcal{F}(\Delta)$ and $\mathcal{F}(\nabla)$ may have infinite type, even if A has finite representation type. In particular, R_A and A do not need to have the same representation type. To see this, we use a result by Dlab and Ringel ([21]).

Proposition 4.2.4. *Let A be the (finite type) algebra $K[x]/\langle x^n\rangle$, $n \in \mathbb{Z}_{>0}$. The subcategories $\mathcal{F}(\Delta)$ and $\mathcal{F}(\nabla)$ of* $\operatorname{mod} R_A$ *have finite type for $n \leq 5$ and infinite type for $n = 6$.*

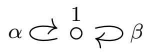
Proof. Let Γ be the Auslander algebra of A, i.e. assume that $\Gamma = \operatorname{End}_A(M)$ where M is the direct sum of all the indecomposable modules in $\operatorname{mod} A$ (up to isomorphism). According to [21, Section 7], the algebra Γ is quasihereditary in a unique way (it is not difficult to see this directly). Moreover, the category $\mathcal{F}(\Delta)$ has finite type for $n \leq 5$ and tubular type E_8 for $n = 6$ ([21, Proposition 7.2]).

Observe that Γ is isomorphic to the algebras $(R_A)^{op}$ and $(S_A)^{op}$, as A is a Nakayama algebra (recall the definition of S_A in Subsection 3.4.1). Since the algebra Γ is quasihereditary in a unique way, then the subcategory $\mathcal{F}(\nabla)$ of $\operatorname{mod} R_A$ has finite type for $n \leq 5$ and infinite type for $n = 6$, and the exact same assertion holds for S_A. Note that $S_A \cong (R_{A^{op}})^{op}$. Since $A \cong A^{op}$ then $S_A \cong (R_A)^{op}$. Because these algebras are quasihereditary in a unique way, then the subcategory $\mathcal{F}(\Delta)$ of $\operatorname{mod} R_A$ also has finite type for $n \leq 5$ and infinite type for $n = 6$. □

Remark 4.2.5. According to [45], the category of ∇-good modules over a right strongly quasihereditary algebra satisfies a tame-wild dichotomy property. Let $(B, \Phi, \sqsubseteq)$ be a right strongly quasiherediatary algebra. By Corollary 4.45 in [45], the subcategory $\mathcal{F}(\nabla)$ of $\operatorname{mod} B$ either has finite type, or it is tame or wild.

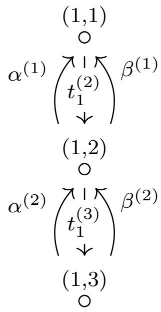
We now give an explicit example of an ADR algebra whose category of ∇-good modules has infinite type. This example was suggested by S. O. Smalø.

Example 4.2.6. Consider the bound quiver algebra $A = KQ/\langle \alpha^2, \beta^2, \alpha\beta - \beta\alpha \rangle$, where

$Q =$  ,

and K is an infinite field.

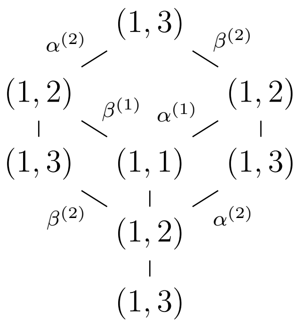
The ADR algebra R_A of A is isomorphic to KQ'/I, where

$Q' =$  ,

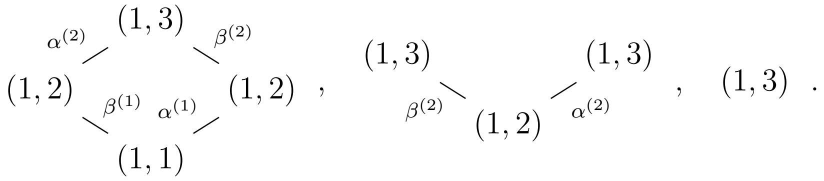
and I is generated by the relations

$$\alpha^{(1)} t_1^{(2)} = \beta^{(1)} t_1^{(2)} = 0,$$
$$t_1^{(2)} \alpha^{(1)} - \alpha^{(2)} t_1^{(3)} = t_1^{(2)} \beta^{(1)} - \beta^{(2)} t_1^{(3)} = 0,$$
$$\alpha^{(1)} \alpha^{(2)} = \beta^{(1)} \beta^{(2)} = \alpha^{(1)} \beta^{(2)} - \beta^{(1)} \alpha^{(2)} = 0.$$

The injective R_A-module $Q_{1,3}$ (which is in fact a projective-injective module) may be represented by

 .

The costandard R_A-modules are given by



Note that $Q_{1,2} \cong Q_{1,3}/L_{1,3}$ (see Proposition 2.5.11). Consider the module $N = Q_{1,2}/L_{1,2}$. We have that $\operatorname{Soc} N = L_{1,3} \oplus L_{1,3} \oplus L_{1,1}$. Given $k \in K$, define $h_k : L_{1,3} \longrightarrow N$ as

$$h_k := \iota \circ \begin{bmatrix} 1_{L_{1,3}} \\ k 1_{L_{1,3}} \\ 0 \end{bmatrix},$$

where ι denotes the inclusion of $\operatorname{Soc} N$ in N. The module $\operatorname{Coker} h_k$, $k \in K$, is indecomposable and it lies in $\mathcal{F}(\nabla)$. It is not difficult to check that $\operatorname{Coker} h_k \ncong \operatorname{Coker} h_{k'}$ for $k \neq k'$. This shows that the category $\mathcal{F}(\nabla)$ has infinite representation type.

4.2.3 Ringel self-duality for ADR algebras

As observed in Subsection 3.4.1, the Ringel dual $\mathcal{R}(R_A)$ of an ADR algebra R_A is somehow similar to the algebra $(R_{A^{op}})^{op}$. It is not unusual for a quasiherederitary algebra $(B, \Phi, \sqsubseteq)$ to be isomorphic to its own Ringel dual $(\mathcal{R}(B), \Phi, \sqsubseteq^{op})$ through a correspondence which preserves the quasihereditary data. This phenomenon is frequently observed in quasihereditary algebras and highest weight categories arising from the theory of semisimple Lie algebras and algebraic groups. Donkin ([23], [24]) and Erdmann–Henke ([28]) have proved that, under certain conditions, a (q-)Schur algebra of type A is its own Ringel dual. Some of these results were extended by Adamovich and Rybnikov ([1]) to Ringel dualities between certain Schur algebras of classical groups. Furthermore, Soergel ([58]) has shown that the category $\mathcal{O}$ is Ringel self-dual.

Therefore, it is natural to ask when the ADR algebra is Ringel self-dual. Recall that R_A is a RUSQ algebra, whereas $\mathcal{R}(R_A)$ is a LUSQ algebra (see Theorem 2.6.1). For $(R_A, \Lambda, \trianglelefteq)$ to be Ringel self-dual it has to be both a RUSQ and LUSQ algebra. According to Erdmann–Parker ([30]) and Ringel ([51]), this implies that $\operatorname{gl.dim} R_A \leq 2$.

Proposition 4.2.7 ([30, §2.1], [51]). *Let $(B, \Phi, \sqsubseteq)$ be a quasihereditary algebra. If B is both left and right strongly quasihereditary then* $\operatorname{gl.dim} B \leq 2$.

By Proposition 2 in [56] (and its proof), if $\mathrm{gl.dim}\, R_A \leq 2$ then $\mathrm{Rad}\, A$ belongs to $\mathrm{add}\, G$. Thus, if the ADR algebra of an Artin algebra A is Ringel self-dual, then $\mathrm{Rad}\, A \in \mathrm{add}\, G$. This shows that Ringel self-dual ADR algebras are quite uncommon. It is an interesting exercise to check that R_A is Ringel self-dual for $A = K[x]/\langle x^n \rangle$, $n \in \mathbb{Z}_{>0}$.

4.2.4 Representation dimension, finitistic dimension and the ADR algebra

The *(little) finitistic dimension* of an Artin algebra A, denoted by $\mathrm{fin.dim}\, A$, is defined to be the supremum of the projective dimensions of all finitely generated modules of finite projective dimension. The finitistic dimension conjecture says that every algebra should have finite finitistic dimension. This conjecture, initially a question by Rosenberg and Zelinski, was published by Bass in 1960 (see [10]) and has attracted the attention of many mathematicians in the last decades.

The representation dimension of an Artin algebra, introduced by Auslander in [4], is also a homological dimension which has been widely studied since its first appearance in the literature. Given an Artin algebra A, the *representation dimension* of A, denoted by $\mathrm{rep.dim}\, A$, is the minimal possible global dimension of the endomorphism algebra of an A-module which is both a generator and a cogenerator of $\mathrm{mod}\, A$.

Example 4.2.8. Take A to be a self-injective Artin algebra. Then the generator G (as in (2.1)) is also a cogenerator of $\mathrm{mod}\, A$. Hence $\mathrm{rep.dim}\, A \leq \mathrm{gl.dim}\, R_A \leq \mathrm{LL}(A)$ (see Corollary 3.3.12).

All the classes of algebras for which Auslander determined the precise representation dimension turned out to have representation dimension at most 3. Thus, he asked whether the representation dimension can be greater than 3, but also, whether it always has to be finite.

One major step towards the understanding of these homological dimensions was made in 2005 by Igusa–Todorov (see [38]). They proved that

$$\mathrm{rep.dim}\, A \leq 3 \Rightarrow \mathrm{fin.dim}\, A < \infty.$$

However, it turns out that there are algebras with arbitrarily large representation dimension. Indeed, Rouquier proved in [52] that the exterior algebra $\Lambda(K^m)$ has representation dimension $m + 1$.

More recently, some authors have been trying to adapt the methods of Igusa–Todorov to get new criteria for the finiteness of the finitistic dimension. In [63], Zhang–Zhang proved that for an Artin algebra B we have

$$\operatorname{rep.dim} B \leq 3 \Rightarrow \operatorname{fin.dim} \xi B \xi < \infty,$$

for ξ any idempotent in B, and asked whether every ADR algebra R_A satisfies $\operatorname{rep.dim} R_A \leq 3$. Note that an affirmative answer to this question would imply the finitistic dimension conjecture, as A is an idempotent subalgebra of R_A (modulo Morita equivalence). In [60], Wei refined the work of Zhang–Zhang. In this paper, the author introduces the notion of n-Igusa–Todorov algebra, $n \in \mathbb{Z}_{\geq 0}$, proves that every such algebra has finite finitistic dimension and observes that algebras of representation dimension not greater than 3 are 0-Igusa–Todorov. Wei also proves that the class of 2-Igusa–Todorov algebras is closed under taking idempotent subalgebras. For the definition below, recall that $\Omega(M)$ is the kernel of the projective cover of M in $\operatorname{mod} A$ and that $\Omega^{i+1}(M) := \Omega(\Omega^i(M))$, $\Omega^0(M) := M$.

Definition 4.2.9 ([60])**.** Let A be an Artin algebra and let $n \geq 0$. Then A is said to be *n-Igusa–Todorov* if there exists V in $\operatorname{mod} A$ such that for any M in $\operatorname{mod} A$ there is an exact sequence

$$0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow \Omega^n(M) \longrightarrow 0 \quad , \tag{4.1}$$

with V_1 and V_2 in $\operatorname{add} V$.

It turns out that there are algebras which are not n-Igusa–Todorov for any n. This is an immediate consequence of a result by Rouquier in [52].

Proposition 4.2.10. *Let K be an uncountable field and consider the exterior algebra $\Lambda(K^m)$, with $m \geq 3$. The algebra $\Lambda(K^m)$ is not n-Igusa–Todorov for any $n \geq 0$.*

Proof. By Corollary 4.4 in [52], given a module V in $\operatorname{mod} \Lambda(K^m)$ there is a module M_V in $\operatorname{mod} \Lambda(K^m)$ such that there is no exact sequence

$$0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow M_V \longrightarrow 0 \quad ,$$

with V_1 and V_2 in $\operatorname{add} V$. Since $\Lambda(K^m)$ is a self-injective algebra then it is easy to conclude the class of all n^{th}-syzygies, $\Omega^n(\operatorname{mod} \Lambda(K^m))$, $n \geq 1$, coincides with the class of all modules in $\operatorname{mod} \Lambda(K^m)$ which are not projective. Therefore the algebra $\Lambda(K^m)$ cannot be n-Igusa–Todorov for any $n \geq 1$ (nor for $n = 0$). □

As a consequence, we get that there are ADR algebras with representation dimension greater than 3.

Corollary 4.2.11. *Let K be an uncountable field and suppose that $m \geq 3$. Then the quasihereditary algebra $R_{\Lambda(K^m)}$ is such that* $\operatorname{rep.dim} R_{\Lambda(K^m)} \geq 4$.

Proof. If we had $\operatorname{rep.dim} R_{\Lambda(K^m)} \leq 3$ then $R_{\Lambda(K^m)}$ would be 0-Igusa–Todorov by Proposition 3.1 in [60]. But then $R_{\Lambda(K^m)}$ would be 2-Igusa–Todorov. As $\Lambda(K^m)$ is an idempotent subalgebra of $R_{\Lambda(K^m)}$ then, by Theorem 3.10 in [60], the algebra $\Lambda(K^m)$ would be 2-Igusa–Todorov. This contradicts Proposition 4.2.10. □

4.3 A class of left strongly quasihereditary algebras

The second part of this chapter is concerned with a slightly weaker version of the class of LUSQ algebras. Our conclusions will then be used in Section 5.4 to produce general examples of LUSQ endomorphism algebras and to show that certain quasihereditary algebras appearing in the literature are LUSQ algebras. The main purpose of this section is to derive auxiliary results for Chapter 5.

It is useful to call to mind the definition of a left strongly quasihereditary algebra, given in Section 2.4.

Definition 4.3.1 ([19], [21, Lemma 4.1])**.** A quasihereditary algebra $(B, \Phi, \sqsubseteq)$ is *left strongly quasihereditary* if it satisfies one of the following equivalent conditions:

1. $\nabla(i)/L_i \in \mathcal{F}(\nabla)$ for all $i \in \Phi$;
2. $\mathcal{F}(\nabla)$ is closed under quotients;
3. for all i in Φ the module $\Delta(i)$ has projective dimension at most one;
4. every module in $\mathcal{F}(\Delta)$ has projective dimension at most one;
5. every divisible module (i.e. every module generated by injectives) belongs to $\mathcal{F}(\nabla)$.

Recall that a left strongly quasihereditary algebra $(B, \Phi, \sqsubseteq)$ is said to be *left ultra strongly quasihereditary* (LUSQ, for short) if P_i is a tilting module whenever $\nabla(i)$ is a simple module (see Subsection 2.5.1). In other words, a quasihereditary algebra $(B, \Phi, \sqsubseteq)$ is a LUSQ algebra if the following axioms are satisfied for every $i \in \Phi$:

(A1*) $\nabla(i)/L_i \in \mathcal{F}(\nabla)$;

(A2*) $P_i \in \mathcal{F}(\nabla)$ whenever $\nabla(i)$ is simple.

Let $(B, \Phi, \sqsubseteq)$ be a LUSQ algebra. By dualising the results for RUSQ algebras obtained in Subsection 2.5.1 (see Propositions 2.5.5 and 2.5.6), we deduce the following.

Proposition 4.3.2. *Let $(B, \Phi, \sqsubseteq)$ be a LUSQ algebra. Then $\mathcal{F}(\nabla)$ is closed under factor modules and the costandard modules are uniserial. It is possible to relabel the elements in Φ as*

$$\Phi = \{(i,j) : 1 \leq i \leq n,\ 1 \leq j \leq l_i\}, \tag{4.2}$$

$n, l_i \in \mathbb{Z}_{>0}$, so that the following holds:

1. *$\nabla(i,j)/L_{i,j} = \nabla(i,j-1)$ for $j > 1$, and $\nabla(i,1) = L_{i,1}$;*
2. *$P_{i,1} = T(i,l_i)$;*
3. *for $M \in \mathcal{F}(\nabla)$, the number of costandard modules appearing in a ∇-filtration of M is given by $\sum_{i=1}^{n}[M : L_{i,1}]$*
4. *a module M belongs to $\mathcal{F}(\nabla)$ if and only if* Top M *is a (finite) direct sum of modules of type $L_{i,1}$;*

As seen in Subsection 2.5.2, the RUSQ algebras possess remarkable properties. The injective indecomposable modules over a RUSQ algebra are quotients of indecomposable tilting modules, and all these modules have a unique ∇-filtration (see Theorem 2.5.8 and Proposition 2.5.11). The corresponding results about LUSQ algebras are summarised in the next theorem.

Theorem 4.3.3. *Let $(B, \Phi, \sqsubseteq)$ be a LUSQ algebra. Using the labelling introduced in* (4.2)*, the chain of inclusions*

$$0 \subset P_{i,l_i} \subset \cdots \subset P_{i,j} \subset \cdots \subset P_{i,1} = T(i,l_i),$$

where $P_{i,j}$ is the projective cover of the simple module the label (i,j), is the unique Δ-filtration of $P_{i,1}$. For $1 \leq j < l_i$, the indecomposable tilting module $T(i,j)$ is isomorphic to $P_{i,1}/P_{i,j+1}$.

We are interested in left strongly quasihereditary algebras whose projective indecomposable modules have a unique Δ-filtration.

4.3.1 Definition of weak LUSQ algebra

Fix a quasihereditary algebra $(B, \Phi, \sqsubseteq)$. Assume that the elements in Φ can be labelled as

$$\Phi = \{(i, j) : 1 \leq i \leq n,\ 1 \leq j \leq l_i\},$$

where

$$0 \subset P_{i,l_i} \subset \ldots \subset P_{i,2} \subset P_{i,1} \tag{4.3}$$

is a Δ-filtration of $P_{i,1}$ (so $P_{i,j}/P_{i,j+1} \cong \Delta(i, j)$ for $1 \leq j < l_i$ and $P_{i,l_i} \cong \Delta(i, l_i)$). As a consequence, every projective indecomposable B-module has a unique Δ-filtration, and any two projective indecomposable modules $P_{i,1}$ and $P_{k,1}$ with $i \neq k$ have no common factors in their Δ-filtration.

According to Theorem 4.3.3, every LUSQ algebra fits into this setup. For this reason, and for ease of reference, we shall call $(B, \Phi, \sqsubseteq)$ a *weak LUSQ* algebra (WLUSQ algebra, for short).

The purpose of this section is to derive the basic properties of WLUSQ algebras. The notion of a WLUSQ algebra may seem a bit artificial and unmotivated right now. Its usefulness will hopefully become apparent in Chapter 5, when investigating strongly quasihereditary endomorphism algebras.

4.3.2 Properties of weak LUSQ algebras

Lemma 4.3.4. *Let $(B, \Phi, \sqsubseteq)$ be a WLUSQ algebra. For every $(i, j) \in \Phi$, there are short exact sequences*

$$0 \longrightarrow P_{i,j+1} \longrightarrow P_{i,j} \longrightarrow \Delta(i, j) \longrightarrow 0 \quad , \tag{4.4}$$

where $P_{i,l_i+1} := 0$. In particular, $(i, j) \sqsubset (i, j')$ for $j < j'$, and the algebra B is left strongly quasihereditary (so $\mathcal{F}(\nabla)$ is closed under quotients).

Proof. The existence of the short exact sequence (4.4) follows from (4.3). The fact that $(i, j) \sqsubset (i, j')$ for $j < j'$ follows from the characterisation of quasihereditary algebras in Proposition 1.4.12. The exact sequences (4.4) imply that every standard B-module has injective dimension at most 1. That is, B is left strongly quasihereditary. □

The previous result shows that there is a chain of class inclusions

LUSQ algebras ⊂ WLUSQ algebras ⊂ left strongly quasihereditary algebras.

The next lemma gives an alternative characterisation of the projective indecomposable modules over a WLUSQ algebra. This result will be used in the proof of Theorem 4.3.13, which describes the tilting modules over a WLUSQ algebra.

Lemma 4.3.5. *Let $(B,\Phi,\sqsubseteq)$ be a WLUSQ algebra. Then*

$$P_{i,j+1}=\mathrm{Tr}\left(\bigoplus_{(k,l):\,(k,l)\not\sqsubseteq(i,j)}P_{k,l},P_{i,p}\right)=I_{i,j}P_{i,p},$$

for all p and j such that $1\leq p\leq j<l_i$, where $I_{i,j}=\mathrm{Tr}(\bigoplus_{(k,l):\,(k,l)\not\sqsubseteq(i,j)}P_{k,l},B)$ is an idempotent ideal in B. In particular, $P_{i,j+1}$ is the largest submodule of $P_{i,p}$ generated by projectives $P_{k,l}$, with $(k,l)\not\sqsubseteq(i,j)$.

Proof. Lemma 4.3.4 implies that $(i,j+1)\not\sqsubseteq(i,j)$, so $P_{i,j+1}$ is a submodule of $P_{i,p}$ generated by $\bigoplus_{(k,l):\,(k,l)\not\sqsubseteq(i,j)}P_{k,l}$. Hence

$$P_{i,j+1}\subseteq\mathrm{Tr}\left(\bigoplus_{(k,l):\,(k,l)\not\sqsubseteq(i,j)}P_{k,l},P_{i,p}\right)=I_{i,j}P_{i,p},$$

where $I_{i,j}$ is the idempotent ideal $\mathrm{Tr}(\bigoplus_{(k,l):\,(k,l)\not\sqsubseteq(i,j)}P_{k,l},B)$ (see Remark 1.3.12).

To prove the other inclusion, note that $P_{i,p}/I_{i,j}P_{i,p}$ is the largest factor module of $P_{i,p}$ whose composition factors are all of the form $L_{k,l}$, with $(k,l)\sqsubseteq(i,j)$ (see Remark 1.4.1). The module $P_{i,p}/P_{i,j+1}$ has a (unique) filtration by the standard modules, with factors $\Delta(i,p),\Delta(i,p+1),\ldots,\Delta(i,j)$. In particular, all the compositions factors of $P_{i,p}/P_{i,j+1}$ are of the form $L_{k,l}$, with $(k,l)\sqsubseteq(i,j)$. So there is an epic $\varepsilon:P_{i,p}/I_{i,j}P_{i,p}\longrightarrow P_{i,p}/P_{i,j+1}$. The inclusion $P_{i,j+1}\subseteq I_{i,j}P_{i,p}$ induces a canonical epic $\pi:P_{i,p}/P_{i,j+1}\longrightarrow P_{i,p}/I_{i,j}P_{i,p}$. So the endomorphism $\varepsilon\circ\pi$ is an epic, and hence it is an automorphism. As a consequence, the map π is injective, which implies that $P_{i,j+1}=I_{i,j}P_{i,p}$. □

We now initiate the study of the costandard modules over a WLUSQ algebra. The chain (4.3) provides information about the structure of the costandard modules.

Lemma 4.3.6. *For every $(i,j)\in\Phi$ we have that*

$$\dim_{\mathrm{End}_B(\nabla(i,j))}\mathrm{Hom}_B\left(P_{k,l},\nabla(i,j)\right)=\begin{cases}1 & \text{if } k=i \text{ and } l\leq j,\\ 0 & \text{otherwise.}\end{cases}$$

The module $\nabla(i,j)$ has composition factors $L_{i,j},L_{i,j-1},\ldots,L_{i,1}$, with $L_{i,j}$ having multiplicity one in $\nabla(i,j)$. In particular, the modules $\nabla(i,1)$, $1\leq i\leq n$, are all the simple costandard modules.

Proof. By the Brauer-Humphreys reciprocity (see Lemma 1.4.11), we have that

$$(P_{k,l} : \Delta(i,j)) = \dim_{\mathrm{End}_B(\nabla(i,j))} \mathrm{Hom}_B(P_{k,l}, \nabla(i,j)).$$

The claim then follows from Lemma 4.3.4. □

The costandard modules over a LUSQ algebra are uniserial and satisfy the identity $\nabla(i,j)/L_{i,j} \cong \nabla(i,j-1)$ (see Proposition 4.3.2). As we shall see shortly, this property does not generally hold for WLUSQ algebras.

Let B be a WLUSQ algebra (over the ring of scalars C), and denote the (Jordan–Hölder) length of a module M in $\mathrm{mod}\, C$ by $l(M)$. Suppose that $l(\mathrm{End}_B(L_{i,j})) = l(\mathrm{End}_B(L_{i,j'}))$ for all i, j and j'. This holds, for instance, when the ring of scalars is an algebraically closed field, or when B is a path algebra or a bound quiver algebra over some field. In this specific situation, the constandard B-modules satisfy the identity $\nabla(i,j)/L_{i,j} \cong \nabla(i,j-1)$. In particular, the costandard modules are uniserial in this case.

Lemma 4.3.7. *Let $(B, \Phi, \sqsubseteq)$ be a WLUSQ algebra (over C). Suppose that the C-modules $\mathrm{End}_B(L_{i,j-1})$ and $\mathrm{End}_B(L_{i,j})$ have the same length, for some $(i,j) \in \Phi$. Then $\nabla(i,j)/L_{i,j} \cong \nabla(i,j-1)$.*

The following remark is used in the proof of Lemma 4.3.7.

Remark 4.3.8. For $x, y \in \Phi$, the multiplicity of L_x in $\mathrm{Top}(\mathrm{Rad}\, P_y)$ coincides with the dimension of $\mathrm{Ext}^1_B(L_y, L_x)$ over the division algebra $\mathrm{End}_B(L_x)$ (see Proposition 1.2.3). Similarly, the multiplicity of L_y in $\mathrm{Soc}(Q_x/L_x)$ coincides with the dimension of $\mathrm{Ext}^1_B(L_y, L_x)$ over $\mathrm{End}_B(L_y)^{op}$. If B is an Artin C-algebra and $l(\mathrm{End}_B(L_x)) = l(\mathrm{End}_B(L_y))$, then

$$\begin{aligned}
\dim_{\mathrm{End}_B(L_x)} \mathrm{Ext}^1_B(L_y, L_x) &= l\left(\mathrm{Ext}^1_B(L_y, L_x)\right)/l(\mathrm{End}_B(L_x)) \\
&= l\left(\mathrm{Ext}^1_B(L_y, L_x)\right)/l(\mathrm{End}_B(L_y)) \\
&= \dim_{\mathrm{End}_B(L_y)^{op}} \mathrm{Ext}^1_B(L_y, L_x),
\end{aligned}$$

so the multiplicity of L_y in $\mathrm{Soc}(Q_x/L_x)$ coincides with the multiplicity of L_x in $\mathrm{Top}(\mathrm{Rad}\, P_y)$.

Proof of Lemma 4.3.7. Since $l(\mathrm{End}_B(L_{i,j-1})) = l(\mathrm{End}_B(L_{i,j}))$, then

$$[\mathrm{Soc}(Q_{i,j}/L_{i,j}) : L_{i,j-1}] = [\mathrm{Top}(\mathrm{Rad}\, P_{i,j-1}) : L_{i,j}] \tag{4.5}$$

by Remark 4.3.8.

Let $p \leq j$. Then $(i,p) \sqsubseteq (i,j)$ by Lemma 4.3.4, so

$$[\mathrm{Soc}(Q_{i,j}/L_{i,j}) : L_{i,p}] = [\mathrm{Soc}(\nabla(i,j)/L_{i,j}) : L_{i,p}]. \tag{4.6}$$

From (4.4) (see Lemma 4.3.4) we get the short exact sequence

$$0 \longrightarrow P_{i,p+1} \longrightarrow \mathrm{Rad}\, P_{i,p} \longrightarrow \mathrm{Rad}\, \Delta(i,p) \longrightarrow 0$$

and from the fact that $(i,j) \sqsupseteq (i,p)$ when $p \leq j$, we conclude that

$$(p \leq j \wedge [\mathrm{Top}(\mathrm{Rad}\, P_{i,p}) : L_{i,j}] \neq 0) \Rightarrow (p+1 = j \wedge [\mathrm{Top}(\mathrm{Rad}\, P_{i,p}) : L_{i,j}] = 1)\,. \tag{4.7}$$

From Lemma 4.3.6, it follows that $[\mathrm{Soc}(\nabla(i,j)/L_{i,j}) : L_{i,p}] \neq 0$ for some $p \leq j$, and all the summands of $\mathrm{Soc}(\nabla(i,j)/L_{i,j})$ must be of the form $L_{i,p}$ with $p \leq j$. The identity (4.6), together with Proposition 1.2.3, implies that $[\mathrm{Top}(\mathrm{Rad}\, P_{i,p}) : L_{i,j}] \neq 0$. But then $p+1 = j$ and $[\mathrm{Top}(\mathrm{Rad}\, P_{i,j-1}) : L_{i,j}] = 1$ by (4.7). Using (4.5) and (4.6), we conclude that

$$[\mathrm{Top}(\mathrm{Rad}\, P_{i,j-1}) : L_{i,j}] = [\mathrm{Soc}(Q_{i,j}/L_{i,j}) : L_{i,j-1}] = [\mathrm{Soc}(\nabla(i,j)/L_{i,j}) : L_{i,j-1}] = 1.$$

This means that $\nabla(i,j)/L_{i,j}$ has simple socle $L_{i,j-1}$. As $\nabla(i,j)/L_{i,j} \in \mathcal{F}(\nabla)$ ($\mathcal{F}(\nabla)$ is closed under quotients), then $\nabla(i,j-1)$ appears in the bottom of a ∇-filtration of $\nabla(i,j)/L_{i,j}$. Note that $(i,j-1)$ is a highest weight in $\nabla(i,j)/L_{i,j}$ (see Lemma 4.3.6). But then Lemma 1.4.5 implies that $\nabla(i,j)/L_{i,j} \cong \nabla(i,j-1) \oplus X$, $X \in \mathcal{F}(\nabla)$. Since $\nabla(i,j)/L_{i,j}$ has simple socle, we must have $\nabla(i,j)/L_{i,j} \cong \nabla(i,j-1)$. □

As an immediate consequence of the previous result, we conclude the following.

Corollary 4.3.9. *Let $(B, \Phi, \sqsubseteq)$ be a WLUSQ algebra (over C). Suppose that the C-modules $\mathrm{End}_B(L_{i,j})$ and $\mathrm{End}_B(L_{i,j'})$ have the same length for all i, j and j' satisfying $1 \leq i \leq n$, $1 \leq j, j'$ and $j, j' \leq l_i$. Then $\nabla(i,j)/L_{i,j} \cong \nabla(i,j-1)$ for every $(i,j) \in \Phi$ (where $\nabla(i,0) := 0$). In particular, the costandard B-modules are uniserial. The composition factors of $\nabla(i,j)$ are the simple modules $L_{i,j}, L_{i,j-1} \ldots, L_{i,1}$, ordered from the socle to the top.*

The costandard modules over a WLUSQ algebra are not necessarily uniserial. The next example illustrates this claim.

Example 4.3.10. Consider the finite-dimensional $\mathbb{R}$-algebra B,

$$B = \begin{bmatrix} \mathbb{C} & \mathbb{C} \\ 0 & \mathbb{R} \end{bmatrix}.$$

The module

$$P_{1,2} := \begin{bmatrix} \mathbb{C} & 0 \\ 0 & 0 \end{bmatrix}$$

is a projective simple B-module. This is 2-dimensional over $\mathbb{R}$.

Similarly, the module

$$P_{1,1} := \begin{bmatrix} 0 & \mathbb{C} \\ 0 & \mathbb{R} \end{bmatrix}$$

is also projective indecomposable over B, and

$$X = \begin{bmatrix} 0 & \mathbb{C} \\ 0 & 0 \end{bmatrix}$$

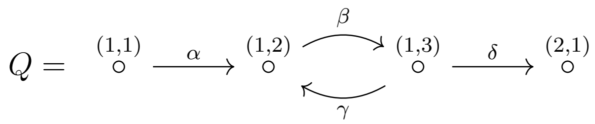
is a B-submodule of $P_{1,1}$ which is isomorphic to $P_{1,2}$. We identify X with $P_{1,2}$. Now $P_{1,1}/P_{1,2} = L_{1,1}$ is 1-dimensional over $\mathbb{R}$, hence simple. Note that B is quasihereditary with respect to the order $(1,1) \sqsubset (1,2)$ and $0 \subset P_{1,2} \subset P_{1,1}$ is the unique Δ-filtration of $P_{1,1}$. Thus B is a WLUSQ algebra with respect to the labelling chosen. By Proposition 1.2.3, the multiplicity of $L_{1,1}$ in $\mathrm{Soc}(Q_{1,2}/L_{1,2})$ equals the dimension of $\mathrm{Ext}^1_B(L_{1,1}, L_{1,2})$ over $\mathrm{End}_B(L_{1,1})^{op} \cong \mathbb{R}$. Now

$$\begin{aligned} \dim_{\mathbb{R}} \mathrm{Ext}^1_B(L_{1,1}, L_{1,2}) &= \dim_{\mathrm{End}_B(L_{1,2})} \mathrm{Ext}^1_B(L_{1,1}, L_{1,2}) \times \dim_{\mathbb{R}} \mathrm{End}_B(L_{1,2}) \\ &= [\mathrm{Top}(\mathrm{Rad}\, P_{1,1}) : L_{1,2}] \times 2 = 2. \end{aligned}$$

Thus $\mathrm{Soc}(Q_{1,2}/L_{1,2}) = L_{1,1} \oplus L_{1,1} = \mathrm{Soc}(\nabla(1,2)/L_{1,2})$. In particular, $\nabla(1,2)$ is not a uniserial module, even though B is a WLUSQ algebra.

We have just seen that the class of LUSQ algebras is properly contained in the class of WLUSQ algebras, since the costandard modules over a WLUSQ algebra are not necessarily uniserial. The tilting modules over a LUSQ algebra are specially nice. We shall now look at the tilting modules over a WLUSQ algebra $(B, \Phi, \sqsubseteq)$. Observe that $\nabla(i,1) \cong L_{i,1}$, and these are all the simple costandard modules over B (see Lemma 4.3.6). For B to be a LUSQ algebra, $P_{i,1}$ would have to be isomorphic to the tilting module $T(i, l_i)$ (see Theorem 4.3.3). This is not usually the case.

Example 4.3.11. Consider the quiver



and the algebra $B = KQ/\langle \beta\gamma \rangle$, where K is a field. Observe that B is quasihereditary with respect to the order $\sqsubseteq$, defined by

$$(k,l) \sqsubset (i,j) \Leftrightarrow l < j.$$

We have that $P_{2,1} = L_{2,1} = \Delta(2,1)$ and $0 \subset P_{1,3} \subset P_{1,2} \subset P_{1,1}$ is the unique Δ-filtration of $P_{1,1}$. That is, B is a WLUSQ algebra with respect to the labelling chosen for the vertices of Q. The projective indecomposable B-modules are given by

$$P_{2,1} = \quad (2,1) \quad , \quad P_{1,1} = \begin{array}{ccccc} & & (1,1) & & \\ & & \mid & & \\ & & (1,2) & & \\ & & \mid & & \\ & & (1,3) & & \\ & \diagup & & \diagdown & \\ (2,1) & & & & (1,2) \end{array} \quad ,$$

$$P_{1,2} = \begin{array}{ccccc} & & (1,2) & & \\ & & \mid & & \\ & & (1,3) & & \\ & \diagup & & \diagdown & \\ (2,1) & & & & (1,2) \end{array} \quad , \quad P_{1,3} = \begin{array}{ccccc} & & (1,3) & & \\ & \diagup & & \diagdown & \\ (2,1) & & & & (1,2) \end{array} \quad .$$

The injective indecomposable modules can be represented as

$$Q_{1,1} = \quad (1,1) \quad , \quad Q_{1,2} = \begin{array}{ccccc} & & & & (1,1) \\ & & & & \mid \\ & & & & (1,2) \\ & & & & \mid \\ (1,1) & & & & (1,3) \\ & \diagdown & & \diagup & \\ & & (1,2) & & \end{array} \quad ,$$

$$Q_{1,3} = \begin{array}{c} (1,1) \\ \mid \\ (1,2) \\ \mid \\ (1,3) \end{array} \quad , \quad Q_{2,1} = \begin{array}{c} (1,1) \\ \mid \\ (1,2) \\ \mid \\ (1,3) \\ \mid \\ (2,1) \end{array} \quad .$$

As expected (see Corollary 4.3.9), the costandard B-modules are uniserial and satisfy the identity $\nabla(i,j)/L_{i,j} \cong \nabla(i,j-1)$. The indecomposable tilting modules are given by

$$T(2,1) = \quad (2,1) \quad , \quad T(1,1) = \quad (1,1) \quad ,$$

$$T(1,2) = \begin{array}{c} (1,1) \\ | \\ (1,2) \end{array} \quad , \quad T(1,3) = \begin{array}{ccccc} & & (1,1) & & \\ & & | & & \\ & & (1,2) & & \\ & & | & & \\ (1,1) & & (1,3) & & \\ & \diagdown \quad \diagup & & \diagdown & \\ & (1,2) & & & (2,1) \end{array} .$$

Observe that $P_{1,1}$ is not a tilting module, so B is not a LUSQ algebra. To be precise, note that $P_{1,1} \ncong T(1,3)$. However, $P_{1,1}$ can be embedded in $T(1,3)$, and $T(1,3)/P_{1,1}$ lies in $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$. We shall see (in Theorem 4.3.13) that this illustrates a general property of WLUSQ algebras.

Each indecomposable tilting module appears as a central term in two special short exact sequences (see Theorem 1.4.14). The next result describes the corresponding short exact sequences (1.4) for WLUSQ algebras.

Proposition 4.3.12. *Let $(B, \Phi, \sqsubseteq)$ be a WLUSQ algebra. For every $(i,j) \in \Phi$ there is a short exact sequence*

$$0 \longrightarrow \Delta(i,j) \xrightarrow{\ \phi\ } T(i,j) \longrightarrow T(i,j-1) \oplus U(i,j) \longrightarrow 0 \ , \qquad (4.8)$$

with $U(i,j) \in \mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$ and $T(i,0) := 0$. The morphism ϕ is a left minimal $\mathcal{F}(\nabla)$-approximation of $\Delta(i,j)$.

Proof. Let the maps ϕ and ψ be as in Theorem 1.4.14. Consider the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & & & L_{i,j} & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & Y(i,j) & \longrightarrow & T(i,j) & \xrightarrow{\ \psi\ } & \nabla(i,j) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \exists t} & & \| & & \downarrow{\scriptstyle \pi} & & \\
0 & \longrightarrow & \mathrm{Ker}(\pi \circ \psi) & \longrightarrow & T(i,j) & \xrightarrow{\ \pi\circ\psi\ } & \nabla(i,j)/L_{i,j} & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & L_{i,j} & & & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array} . \qquad (4.9)$$

Using the second row of this diagram, we get a new diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & & & \operatorname{Ker} u & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & \Delta(i,j) & \xrightarrow{\phi} & T(i,j) & \longrightarrow & X(i,j) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \exists v} & & \Big\| & & \Big\downarrow{\scriptstyle \exists u} & & \\
0 & \longrightarrow & \operatorname{Ker}(\pi\circ\psi) & \longrightarrow & T(i,j) & \xrightarrow{\pi\circ\psi} & \nabla(i,j)/L_{i,j} & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & \operatorname{Coker} v & & & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array} . \qquad (4.10)$$

Here the morphisms v and u exist because $\pi \circ \psi \circ \phi = 0$.

We claim that $\operatorname{Ker} u \in \mathcal{F}(\nabla)$. Let ι denote the inclusion of $\operatorname{Rad}\Delta(i,j)$ in $\Delta(i,j)$. We have the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Rad}\Delta(i,j) & \xrightarrow{\iota} & \Delta(i,j) & \longrightarrow & L_{i,j} & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \exists y} & & \Big\downarrow{\scriptstyle v} & & \Big\downarrow{\scriptstyle \exists w} & & \\
0 & \longrightarrow & Y(i,j) & \xrightarrow{t} & \operatorname{Ker}(\pi\circ\psi) & \xrightarrow{\operatorname{coker} t} & L_{i,j} & \longrightarrow & 0
\end{array} .$$

Again, the morphisms y and w exist because $(\operatorname{coker} t) \circ v \circ \iota = 0$. Since the composition factor $L_{i,j}$ appears exactly once in the composition series of $\operatorname{Ker}(\pi \circ \psi)$ (look at the diagram (4.9)), and since $v \neq 0$ (as $\phi \neq 0$), then $(\operatorname{coker} t) \circ v \neq 0$. This implies that $w \neq 0$, hence w is an isomorphism. So

$$\operatorname{Coker} y \cong \operatorname{Coker} v \cong \operatorname{Ker} u.$$

Note that $\operatorname{Coker} y$ lies in $\mathcal{F}(\nabla)$, as $\mathcal{F}(\nabla)$ is closed under quotients by Lemma 4.3.4. Thus $\operatorname{Ker} u \in \mathcal{F}(\nabla)$.

By the same argument (see Lemma 4.3.4), the module $\nabla(i,j)/L_{i,j}$ lies in $\mathcal{F}(\nabla)$. Consequently, the module $X(i,j)$ belongs to $\mathcal{F}(\nabla) \cap \mathcal{F}(\Delta)$ (look at the right hand column of the diagram (4.10)). By Lemma 4.3.6, $L_{i,j-1}$ is the composition factor with highest weight appearing in the composition series of $\nabla(i,j)/L_{i,j}$. By part 2 of Lemma 1.4.5, the factor $\nabla(i,j-1)$ must appear in the top part of a ∇-filtration of $\nabla(i,j)/L_{i,j}$. Now, by looking at the right hand column in (4.10), and by noticing that the modules $\operatorname{Ker} u$, $\nabla(i,j)/L_{i,j}$ lie in $\mathcal{F}(\nabla)$, we conclude that there exists a short exact sequence

$$0 \longrightarrow \operatorname{Ker} z \longrightarrow X(i,j) \xrightarrow{z} \nabla(i,j-1) \longrightarrow 0 \quad ,$$

with $\operatorname{Ker} z \in \mathcal{F}(\nabla)$. The module $X(i,j)$, which lies in $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$, is a direct sum of indecomposable tilting modules. So

$$X(i,j) = \bigoplus_{\theta \in \Theta} T(k_\theta, l_\theta),$$

for some (finite) set of indexes Θ. The factor $\nabla(i,j-1)$ has to be in the top of a ∇-filtration of $T(k_{\theta'}, l_{\theta'})$, for some $\theta' \in \Theta$. Thus $(k_{\theta'}, l_{\theta'}) = (i, j-1)$, for some $\theta' \in \Theta$ (this follows from Lemma 2.5.7, part 3). Therefore, we get the short exact sequence

$$0 \longrightarrow \Delta(i,j) \xrightarrow{\phi} T(i,j) \longrightarrow T(i,j-1) \oplus U(i,j) \longrightarrow 0 \quad ,$$

where $U(i,j) = \bigoplus_{\theta \in \Theta - \{\theta'\}} T(k_\theta, l_\theta)$. □

Let $(B, \Phi, \sqsubseteq)$ be a LUSQ algebra. In this case, the modules $U(i,j)$ appearing in statement of Proposition 4.3.12 are all zero – this is a consequence of Theorem 4.3.3. According to Theorem 4.3.3, the indecomposable tilting B-modules $T(i,j)$ are given by the quotients $P_{i,1}/P_{i,j+1}$. As observed in Example 4.3.11, this does not usually hold for WLUSQ algebras. However, the following property is satisfied.

Theorem 4.3.13. *Let $(B, \Phi, \sqsubseteq)$ be a WLUSQ algebra. For every $(i,j) \in \Phi$ there is a short exact sequence*

$$0 \longrightarrow P_{i,1}/P_{i,j+1} \xrightarrow{\phi_{i,j}} T(i,j) \longrightarrow V(i,j) \longrightarrow 0 \quad ,$$

with $V(i,j) \in \mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$ and $P_{i,l_i+1} := 0$. The map $\phi_{i,j}$ is a left minimal $\mathcal{F}(\nabla)$-approximation of $P_{i,1}/P_{i,j}$.

Proof. To prove the existence of this short exact sequence, we proceed by induction on j, starting with $j = 1$. In this case $P_{i,1}/P_{i,2} \cong \Delta(i,1)$, so by Proposition 4.3.12 there is a short exact sequence

$$0 \longrightarrow P_{i,1}/P_{i,2} \xrightarrow{\phi} T(i,1) \longrightarrow V(i,1) \longrightarrow 0 \quad ,$$

with $V(i,1) := X(i,1) = U(i,1) \in \mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$. Suppose that the claim holds for $(i, j-1) \in \Phi$, where $j \geq 2$. By Proposition 4.3.12, there is a commutative diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & \Delta(i,j) & \longrightarrow & P_{i,1}/P_{i,j+1} & \longrightarrow & P_{i,1}/P_{i,j} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \exists v} & & \downarrow{\scriptstyle \exists u} & & \downarrow{\scriptstyle \begin{bmatrix} \phi_{i,j-1} \\ 0 \end{bmatrix}} & & \\
0 & \longrightarrow & \Delta(i,j) & \xrightarrow{\phi} & T(i,j) & \longrightarrow & T(i,j-1) \oplus U(i,j) & \longrightarrow & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & V(i,j-1) \oplus U(i,j) & & \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array} \quad . \tag{4.11}$$

To prove the existence of the map u, note that the modules $T(i,j)$ and $T(i,j-1) \oplus U(i,j)$ are in $\operatorname{mod}(B/I_{i,j})$, as all its composition factors are of the form $L_{k,l}$ with $(k,l) \sqsubseteq (i,j)$ (recall the notation $I_{i,j}$ introduced in Lemma 4.3.5). Furthermore, observe that $P_{i,1}/P_{i,j+1} = P_{i,1}/I_{i,j}P_{i,1}$ is a projective in $\operatorname{mod}(B/I_{i,j})$ (see Lemma 4.3.5). This proves that u exists. Consequently, there is a morphism v as depicted above, and v is either an automorphism or it is zero.

We claim that v cannot be zero. Suppose, by contradiction, that $v = 0$. Then $\operatorname{Ker} u \cong \operatorname{Ker} v = \Delta(i,j)$, and $\operatorname{Coker} v \cong \Delta(i,j)$. So $\operatorname{Im} u \cong P_{i,1}/P_{i,j}$, and $\operatorname{Coker} u \in \mathcal{F}(\Delta)$ (as it is an extension of the modules $\Delta(i,j)$ and $V(i,j-1) \oplus U(i,j)$, both in $\mathcal{F}(\Delta)$). Since $\Delta(i,j-1)$ appears in the bottom of a Δ-filtration of $\operatorname{Im} u$, then the short exact sequence

$$0 \longrightarrow \operatorname{Im} u \longrightarrow T(i,j) \longrightarrow \operatorname{Coker} u \longrightarrow 0$$

(where $\operatorname{Im} u$ and $\operatorname{Coker} u$ lie in $\mathcal{F}(\Delta)$) gives rise to a short exact sequence

$$0 \longrightarrow \Delta(i,j-1) \longrightarrow T(i,j) \longrightarrow X \longrightarrow 0 \quad ,$$

with $X \in \mathcal{F}(\Delta)$. This contradicts part 3 of the dual version of Lemma 2.5.7. Thus $v \neq 0$.

Therefore v is an isomorphism and the diagram (4.11) gives rise to the exact sequence

$$0 \longrightarrow P_{i,1}/P_{i,j+1} \xrightarrow{u} T(i,j) \longrightarrow V(i,j) \longrightarrow 0 \ ,$$

where $V(i,j) := V(i,j-1) \oplus U(i,j)$. By induction $V(i,j-1) \in \mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$, and by Proposition 4.3.12 the module $U(i,j)$ lies in $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$. Hence $V(i,j)$ belongs to $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$. Recall the notion of (minimal) approximation introduced in Subsection 1.2.2. The map $\phi_{i,j} := u$ is a left $\mathcal{F}(\nabla)$-approximation because $\mathrm{Ext}^1_B(V(i,j), \mathcal{F}(\nabla)) = 0$ (see Lemma 1.4.8, part 2). Since $T(i,j)$ is an indecomposable module, then $\phi_{i,j}$ is a left minimal morphism. □

Chapter 5

Strongly quasihereditary endomorphism algebras

5.1 Overview of the chapter

Quasihereditary algebras are abundant in mathematics. They often arise as endomorphism algebras of modules endowed with some sort of 'stratification', and in many cases they possess a double centralizer property. Examples coming from the classic realm of semisimple Lie algebras and algebraic groups include the Schur algebras ([34]), and extensions and generalisations of these, like the q-Schur algebras ([17], [24]) and other quasihereditary algebras associated to Hecke algebras ([26]) and to diagram algebras ([36], [37]). There are many more examples of quasihereditary algebras arising from the most diverse contexts.

It has been noticed that certain endomorphism algebras emerging naturally in representation theory are particularly well behaved: they are strongly quasihereditary (recall Definition 4.3.1). Examples of these include: the Auslander algebras, associated to algebras of finite type; the endomorphism algebras constructed by Iyama, used in his famous proof of the finiteness of the representation dimension of Artin algebras ([39]); certain cluster-tilted algebras studied by Geiß–Leclerc–Schröer ([32], [31]) and Iyama–Reiten ([40]). In Chapter 2, we have seen that the ADR algebra R_A of A is yet another example of a strongly quasihereditary endomorphism algebra.

In this chapter, we describe a general way of constructing left strongly quasihereditary endomorphism algebras, and we show that all the examples mentioned in the previous paragraph fit into our setting. Our procedure can be loosely described as follows. One starts with an initial module X_1, which then gives rise to a chain of proper submodules,

$$0 \subset X_m \subset \cdots \subset X_2 \subset X_1,$$

with specific properties. We prove that the algebra $\Gamma = \mathrm{End}_A(\bigoplus_{i=1}^m X_i)^{op}$ is left strongly quasihereditary with global dimension not greater than m. In Section 5.4, we investigate when the algebra Γ is a LUSQ algebra (as defined in Section 4.3). This happens, for instance, if X_1 is an injective module (or, more generally, if X_1 "behaves like" an injective). Therefore, we provide a unified proof to the fact that the cluster-tilted algebras in [32] and the algebras $S_A \cong (R_{A^{op}})^{op}$ (see Subsection 3.4.1) are LUSQ algebras. The methods developed in this chapter can be dualised to produce right strongly quasihereditary algebras and RUSQ algebras.

Throughout this chapter A will denote a C-algebra in the sense of Definition 1.2.1. One should typically think of A as an algebra over a field K (possibly infinite-dimensional). Recall that the category $\mathrm{mod}\, A$, of all A-modules which are finitely generated over C, is a Krull-Schmidt abelian subcategory of $\mathrm{Mod}\, A$, and all the modules in $\mathrm{mod}\, A$ have finite length (see Proposition 1.2.2).

5.2 Construction

In this section, we describe a general construction that produces left strongly quasi-hereditary endomorphism algebras. We are going to add conditions to our setup along the way. In every step, and unless otherwise stated, we shall assume that the conditions introduced up to that point are satisfied.

Our construction relies on a chain of inclusions stemming from an initial module X_1. We need to choose submodules of X_1 so that direct sums are 'preserved'. A way of doing this is by applying a preradical to our original module. Recall that a preradical in $\mathrm{mod}\, A$ is just a subfunctor of the identity functor in $\mathrm{mod}\, A$ (see Definition 1.3.1).

So consider,

(C1) a sequence of preradicals $\tau_{(i)}$ in $\mathrm{mod}\, A$, $1 \leq i \leq m$.

Additionally,

(C2) consider modules X_1 in $\mathrm{mod}\, A$ and $X_{i+1} := \tau_{(i)}(X_i)$, $1 \leq i \leq m$, and suppose that $X_{m+1} = 0$;

(C3) assume that X_{i+1} is a proper submodule of X_i , $1 \leq i \leq m$, and define $\tilde{X} := \oplus_{i=1}^m X_i$, $\Gamma := \mathrm{End}_A(\tilde{X})^{op}$.

In this situation we say that the module $\tilde{X}$ has m *layers*. According to Proposition 1.2.2, Γ is an Artin algebra, and the Γ-modules $\mathrm{Hom}_A(\tilde{X}, M)$, $M \in \mathrm{mod}\, A$, lie in $\mathrm{mod}\, \Gamma$.

Lemma 5.2.1. *Assume (C1)–(C3). Let* $1 \leq i \leq m$ *and suppose that* Y *is a summand of* X_i *which is not a summand of* $X_{i+1} = \tau_{(i)}(X_i)$*. Then* $\tau_{(i)}(Y)$ *is a proper submodule of* Y*.*

Proof. If we had $\tau_{(i)}(Y) = Y$ then Y would be a summand of $\tau_{(i)}(X_i)$, since Y is a summand of X_i and $\tau_{(i)}$ preserves finite direct sums (see Lemma 1.3.3). □

Now

(C4) define $\tilde{X}_{>i} := \bigoplus_{j=i+1}^{m+1} X_j$, $1 \leq i \leq m$; and for Y an indecomposable summand of X_i which is not a summand of X_{i+1}, and Y' an indecomposable module in $\operatorname{add} \tilde{X}_{>i}$, assume that any morphism $g : Y' \longrightarrow Y$ is such that $\operatorname{Im} g \subseteq \tau_{(i)}(Y)$.

For the next proposition recall the definition of right minimal approximation introduced in Subsection 1.2.2.

Proposition 5.2.2. *Assume (C1)–(C4). Let* Y *be an indecomposable summand of* X_i *which is not a summand of* X_{i+1}*, where* $1 \leq i \leq m$*. Then the proper inclusion*

$$\tau_{(i)}(Y) \xhookrightarrow{\iota} Y$$

is a right minimal $\operatorname{add} \tilde{X}_{>i}$*-approximation of* Y*.*

Proof. Any monic is trivially a right minimal morphism. By Lemma 5.2.1, ι is a proper inclusion. Note that $\tau_{(i)}(Y)$ is in $\operatorname{add} \tilde{X}_{>i}$, as $\tau_{(i)}(Y)$ is a summand of $X_{i+1} = \tau_{(i)}(X_i)$ (and this holds because Y is a summand of X_i and $\tau_{(i)}$ preserves direct sums). So it is enough to prove that

$$\operatorname{Hom}_A\left(Y', \tau_{(i)}(Y)\right) \xrightarrow{\iota_*} \operatorname{Hom}_A(Y', Y)$$

is an epic for every indecomposable module Y' in $\operatorname{add} \tilde{X}_{>i}$. Consider an arbitrary morphism $g : Y' \longrightarrow Y$. By assumption (C4), $\operatorname{Im} g \subseteq \tau_{(i)}(Y)$. Thus the morphism g factors through the inclusion ι, and this proves the claim. □

Remark 5.2.3. Let X, X' be in $\operatorname{mod} A$ and suppose we have an inclusion morphism $\iota : X' \longrightarrow X$ which is a right $\operatorname{add} \Theta$-approximation of X for some class Θ of A-modules. Recall the definition of trace introduced in Example 1.3.2. It is not difficult to see that $X' = \operatorname{Tr}(\Theta, X)$. The inclusion $X' \subseteq \operatorname{Tr}(\Theta, X)$ follows from the fact that X' is trivially generated by Θ, as it lies in $\operatorname{add} \Theta$. Moreover, any morphism $f : Y' \longrightarrow X$, with Y' in $\operatorname{add} \Theta$ factors through ι. This proves that $\operatorname{Tr}(\Theta, X) \subseteq X'$.

We apply this observation to our setup. Let Y be an indecomposable summand of X_i which is not a summand of X_{i+1}, $1 \leq i \leq m$. From Proposition 5.2.2, we see that $\tau_{(i)}(Y) = \mathrm{Tr}(\tilde{X}_{>i}, Y)$. This implies that in condition (C1), we may always assume that $\tau_{(i)} = \mathrm{Tr}(\tilde{X}_{>i}, -)$ (but this is sort of an *a posteriori* choice).

As we shall see in Section 5.3, the preradicals $\tau_{(i)}$ will typically be hereditary preradicals in $\operatorname{mod} A$ (as in Definition 1.3.7). Although the preradicals $\tau_{(i)}$ can always be written in the form $\mathrm{Tr}(\Theta, -)$ for a class of modules Θ in $\operatorname{mod} A$, they do not always arise "in nature" in this way. We refer to Appendix B for natural ways of constructing hereditary preradicals.

Corollary 5.2.4. *Assume (C1)–(C4). Let Y be an indecomposable summand of X_i and of X_j, where $1 \leq i < j \leq m$. Then Y is a summand of X_k for $i \leq k \leq j$.*

Proof. Suppose, by contradiction, that Y is not a summand of X_{i+1}. Then, by Proposition 5.2.2, the proper inclusion

$$\tau_{(i)}(Y) \overset{\iota}{\hookrightarrow} Y$$

is a right $\operatorname{add} \tilde{X}_{>i}$-approximation of Y. As Y is also in $\operatorname{add} \tilde{X}_{>i}$, then 1_Y factors through ι, which leads to a contradiction. □

Let Y be an indecomposable summand of $\tilde{X}$. By the previous result, there is a unique i such that Y is a summand of X_i but not a summand of X_{i+1}. Furthermore, note that the number i only depends on the isomorphism class of Y and not on the module Y itself. In this case we say that Y is in the *layer i of $\tilde{X}$*, $1 \leq i \leq m$. We write $ly(Y) = i$.

Let Ψ be a labelling set for the isomorphism classes of indecomposable summands of $\tilde{X}$. Denote the indecomposable summands of $\tilde{X}$ by Y_λ, $\lambda \in \Psi$, and write $ly(\lambda) := ly(Y_\lambda)$. The projective indecomposable Γ-modules are given by $\mathrm{Hom}_A(\tilde{X}, Y_\lambda)$, and we shall denote these by P_λ. Similarly, denote the simple factor of the Γ-module P_λ by L_λ. Define a partial order $\preceq$ on Ψ by setting $\lambda \prec \nu$ for $\lambda, \nu \in \Psi$, if $ly(\lambda) < ly(\nu)$.

Remark 5.2.5. Note that λ and ν are not related in $(\Psi, \preceq)$ if and only if $ly(\lambda) = ly(\nu)$ but $\lambda \neq \nu$, that is, if and only if Y_λ and Y_ν are in the same layer of $\tilde{X}$ but $Y_\lambda \ncong Y_\nu$.

Proposition 5.2.6. *Assume (C1)–(C4). Let Y_λ, $\lambda \in \Psi$, be an indecomposable summand of $\tilde{X}$, and suppose that $ly(\lambda) = i$. Then*

$$\mathrm{Hom}_A(\tilde{X}, \tau_{(i)}(Y_\lambda)) = \mathrm{Tr}\left(\bigoplus_{\nu :\, \nu \succ \lambda} P_\nu, P_\lambda \right).$$

Proof. By Proposition 5.2.2, $\tau_{(i)}(Y_\lambda)$ is in add $\tilde{X}_{>i}$, so $\mathrm{Hom}_A(\tilde{X}, \tau_{(i)}(Y_\lambda))$ is a direct sum of projectives P_ν, with $\nu \succ \lambda$. In particular, $\mathrm{Hom}_A(\tilde{X}, \tau_{(i)}(Y_\lambda))$ is generated by projectives P_ν, $\nu \succ \lambda$. This proves one of the inclusions. In order to prove the other inclusion, consider an arbitrary morphism $g_* : P_\nu \longrightarrow P_\lambda$, with $\nu \succ \lambda$. We have $g_* = \mathrm{Hom}_A(\tilde{X}, g)$ for $g : Y_\nu \longrightarrow Y_\lambda$ (see Proposition 1.2.4), with Y_λ in the layer i of $\tilde{X}$ and Y_ν in the layer j of $\tilde{X}$, for some $j > i$. By (C4), we must have $\mathrm{Im}\, g \subseteq \tau_{(i)}(Y_\lambda)$. Thus

$$\mathrm{Im}\, g_* \subseteq \mathrm{Hom}_A(\tilde{X}, \mathrm{Im}\, g) \subseteq \mathrm{Hom}_A(\tilde{X}, \tau_{(i)}(Y_\lambda)).$$

The inclusion $\mathrm{Tr}(\bigoplus_{\nu:\, \nu \succ \lambda} P_\nu, P_\lambda) \subseteq \mathrm{Hom}_A(\tilde{X}, \tau_{(i)}(Y_\lambda))$ follows from this last observation. □

From now onwards, suppose additionally that

(C5) for any nonisomorphic indecomposable summands Y_λ, Y_ν of $\tilde{X}$, with $i = ly(\lambda) = ly(\nu)$, every morphism $g : Y_\lambda \longrightarrow Y_\nu$ is such that $\mathrm{Im}\, g \subseteq \tau_{(i)}(Y_\nu)$.

We now recall the definition of a standardly stratified algebra (see for instance [21]). Let B be an Artin C-algebra, and assume that $(\Phi, \sqsubseteq)$ is a labelling poset for the isomorphism classes of simple B-modules. Denote the projective indecomposable B-modules by P_i, $i \in \Phi$. Let L_i be the simple factor of P_i, and let $\Delta(i)$ be the standard B-module with label $i \in \Phi$, as defined in (1.2), Subsection 1.4.1. We say that B is *standardly stratified* with respect to $(\Phi, \sqsubseteq)$ if, for every $i \in \Phi$,

1. $P_i \in \mathcal{F}(\Delta)$;
2. $(P_i : \Delta(i)) = 1$ and $(P_i : \Delta(j)) \neq 0$ implies that $j \sqsupseteq i$.

According to Proposition 1.4.12, a standardly stratified algebra $(B, \Phi, \sqsubseteq)$ is quasihereditary if $[\Delta(i) : L_i] = 1$ for all $i \in \Phi$. Moreover, a standardly stratified algebra is quasihereditary if and only if it has finite global dimension (see [62]; see also [44] for a similar result).

Proposition 5.2.7. *Assume (C1)–(C5). The algebra* Γ *is standardly stratified with respect to* $(\Psi, \preceq)$*. Moreover, if the module* Y_λ, $\lambda \in \Psi$, *is such that* $ly(\lambda) = i$*, then*

$$\mathrm{Hom}_A(\tilde{X}, \tau_{(i)}(Y_\lambda)) = \mathrm{Tr}\left(\bigoplus_{\nu:\, \nu \npreceq \lambda} P_\nu, P_\lambda\right),$$

and this module is a direct sum of projective Γ*-modules* P_ν*, with* $\nu \succ \lambda$*. In particular,*

$$\Delta(\lambda) = P_\lambda / \mathrm{Hom}_A(\tilde{X}, \tau_{(i)}(Y_\lambda)),$$

and $\mathrm{proj.\,dim}\, \Delta(\lambda) \leq 1$.

Proof. We start by proving that

$$\operatorname{Hom}_A(\tilde{X}, \tau_{(i)}(Y_\lambda)) = \operatorname{Tr}\left(\bigoplus_{\nu:\,\nu\not\preceq\lambda} P_\nu, P_\lambda\right). \tag{5.1}$$

By Proposition 5.2.6, it is enough to show that for a morphism $g_* : P_\nu \longrightarrow P_\lambda$, with ν and λ not related in $(\Psi, \preceq)$, we have that $\operatorname{Im} g_* \subseteq \operatorname{Hom}_A(\tilde{X}, \tau_{(i)}(Y_\lambda))$. Note that for such a map g_*, we have $g_* = \operatorname{Hom}_A(\tilde{X}, g)$, with $g : Y_\nu \longrightarrow Y_\lambda$, where $ly(\lambda) = ly(\nu)$, but $Y_\lambda \not\cong Y_\nu$ (see Remark 5.2.5). By assumption (C5), $\operatorname{Im} g$ is contained in $\tau_{(i)}(Y_\lambda)$. So

$$\operatorname{Im} g_* \subseteq \operatorname{Hom}_A(\tilde{X}, \operatorname{Im} g) \subseteq \operatorname{Hom}_A(\tilde{X}, \tau_{(i)}(Y_\lambda)),$$

and the identity (5.1) holds.

Thus, by the definition of standard module, we have a short exact sequence

$$0 \longrightarrow \operatorname{Hom}_A(\tilde{X}, \tau_{(i)}(Y_\lambda)) \xrightarrow{\iota_*} P_\lambda \longrightarrow \Delta(\lambda) \longrightarrow 0 \ .$$

Suppose $\tilde{X}$ has m layers. If $i = m$, i.e. if Y_λ is in the layer m of $\tilde{X}$, then $\tau_{(m)}(Y_\lambda) = 0$ as this is a summand of $\tau_{(m)}(X_m) = X_{m+1} = 0$ (see (C2)). So we have $P_\lambda = \Delta(\lambda)$ in this case, and P_λ satisfies conditions 1 and 2 in the definition of a standardly stratified algebra. Suppose conditions 1 and 2 are satisfied for the modules P_ν, with $ly(\nu) > i = ly(\lambda)$. By Proposition 5.2.2, $\operatorname{Hom}_A(\tilde{X}, \tau_{(i)}(Y_\lambda))$ is a direct sum of modules P_ν, with $ly(\nu) > i$. But then P_λ satisfies conditions 1 and 2 in the definition of a standardly stratified algebra. This concludes the proof. □

Finally, assume that the following holds:

(C6) any nonisomorphism $g : Y_\lambda \longrightarrow Y_\lambda$, with $ly(\lambda) = i$, is such that $\operatorname{Im} g \subseteq \tau_{(i)}(Y_\lambda)$.

Recall that a quasihereditary algebra $(B, \Phi, \sqsubseteq)$ is left strongly quasihereditary if every standard module has projective dimension at most 1, or equivalently, if $\mathcal{F}(\nabla)$ is closed under factor modules (see Definition 4.3.1).

Theorem 5.2.8. *Assume (C1)–(C6). The algebra* Γ *is quasihereditary with respect to* $(\Psi, \preceq)$*. In fact,* $(\Gamma, \Psi, \preceq)$ *is a left strongly quasihereditary algebra and* $\operatorname{proj.dim} L_\lambda \leq ly(\lambda)$ *for every* $\lambda \in \Psi$*. In particular,* $\operatorname{gl.dim} \Gamma \leq m$*, where* m *is the number of layers of* $\tilde{X}$*.*

Proof. By Proposition 5.2.7, $(\Gamma, \Psi, \preceq)$ is a standardly stratified algebra, and we have short exact sequences

$$0 \longrightarrow \operatorname{Hom}_A(\tilde{X}, \tau_{(i)}(Y_\lambda)) \xrightarrow{\iota_*} P_\lambda \longrightarrow \Delta(\lambda) \longrightarrow 0 \quad , \tag{5.2}$$

where $ly(\lambda) = i$, and $\operatorname{Hom}_A(\tilde{X}, \tau_{(i)}(Y_\lambda))$ is a direct sum of projective Γ-modules P_ν, with $\nu \succ \lambda$. In particular the standard Γ-modules have projective dimension not greater than 1. Consider a nonisomorphism $g_* : P_\lambda \longrightarrow P_\lambda$. Note that $g_* = \operatorname{Hom}_A(\tilde{X}, g)$, where $g : Y_\lambda \longrightarrow Y_\lambda$ is a nonisomorphism (see Proposition 1.2.4). By assumption (C6), we have $\operatorname{Im} g \subseteq \tau_{(i)}(Y_\lambda)$, so $\operatorname{Im} g_* \subseteq \operatorname{Hom}_A(\tilde{X}, \tau_{(i)}(Y_\lambda))$. Thus $\operatorname{Rad}\Delta(\lambda)$ has no composition factors of the form L_λ. This proves that $(\Gamma, \Psi, \preceq)$ is a left strongly quasihereditary algebra.

We now prove that $\operatorname{proj.dim} L_\lambda \leq i$, where $i = ly(\lambda)$. Note that the desired upper bound for the global dimension of Γ will then follow by inductively applying this claim and the Horseshoe Lemma (see [61, 2.2.8]). So suppose $i = 1$, i.e. assume that Y_λ is in layer 1 of $\tilde{X}$. In this case we have $\Delta(\lambda) = L_\lambda$, since λ is minimal in $(\Psi, \preceq)$ (and Γ is quasihereditary). The exact sequence (5.2) implies that $\operatorname{proj.dim} L_\lambda \leq 1$ in this case. Suppose now that $ly(\lambda) = i \geq 2$, and assume that $\operatorname{proj.dim} L_\nu \leq ly(\nu)$, for every ν such that $ly(\nu) < i$. Since all composition factors of $\operatorname{Rad}\Delta(\lambda)$ are of the form L_ν with $\nu \prec \lambda$ (i.e. with $ly(\nu) < ly(\lambda) = i$), we have that $\operatorname{proj.dim} \operatorname{Rad}\Delta(\lambda) \leq i-1$ by the induction hypothesis (using the Horseshoe Lemma). Note that the exact sequence (5.2) gives rise to the exact sequence

$$0 \longrightarrow \operatorname{Hom}_A(\tilde{X}, \tau_{(i)}(Y_\lambda)) \longrightarrow \operatorname{Rad} P_\lambda \longrightarrow \operatorname{Rad}\Delta(\lambda) \longrightarrow 0 \quad , \tag{5.3}$$

where $\operatorname{Hom}_A(\tilde{X}, \tau_{(i)}(Y_\lambda))$ is a projective module according to Proposition 5.2.7. Since $\operatorname{proj.dim} \operatorname{Rad}\Delta(\lambda) \leq i-1$, then $\operatorname{proj.dim} \operatorname{Rad} P_\lambda \leq i-1$ by the Horseshoe Lemma. Hence $\operatorname{proj.dim} L_\lambda \leq i = ly(\lambda)$. □

5.3 Applications

The algebra S_A introduced in Subsection 3.4.1, the quasihereditary algebras described by Iyama in [39], and the cluster-tilted algebras studied in [32], [31] and [40] are all examples of left strongly quasihereditary algebras. In this section we shall see that these examples fit into the construction outlined in Section 5.2.

5.3.1 The ADR algebra and generalisations

Let X_1 be in $\operatorname{mod} A$, and assume that any pair of nonisomorphic indecomposable summands of X_1 have nonisomorphic simple socle. Without loss of generality we may suppose that X_1 is basic, i.e. we may assume that $X_1 = \bigoplus_{j=1}^{n} Y_j$ where each Y_j has simple socle and $\operatorname{Soc} Y_j \ncong \operatorname{Soc} Y_k$ for $j \neq k$.

Suppose that X_1 has Loewy length m and define $\tau_{(i)} := \operatorname{Soc}_{m-i}(-)$, $1 \leq i \leq m$. Using this data and the notation in Section 5.2 observe that assumptions (C1), (C2) and (C3) are satisfied. In this particular case $\tilde{X}$ has m layers.

In order to check that assumption (C4) holds, let Y be an indecomposable summand of X_i which is not a summand of X_{i+1}, and let Y' be an indecomposable summand of X_j, for some $j > i$. I.e., suppose that $\operatorname{LL}(Y) = m - i + 1$ and that $\operatorname{LL}(Y') < m - i + 1$. Given a map $g : Y' \longrightarrow Y$ we have that $\operatorname{LL}(\operatorname{Im} g) < m - i + 1$, so $\operatorname{Im} g \subseteq \operatorname{Soc}_{m-i} Y = \tau_{(i)}(Y)$.

To show that assumption (C5) is satisfied, consider a morphism $g : Y_\lambda \longrightarrow Y_\nu$, with $ly(\lambda) = ly(\nu) = i$ but $Y_\lambda \ncong Y_\nu$. This implies that both Y_λ and Y_ν have Loewy length $m - i + 1$, but they have distinct simple socle. In this situation we must have $\operatorname{LL}(\operatorname{Im} g) < m - i + 1$, thus $\operatorname{Im} g \subseteq \operatorname{Soc}_{m-i} Y = \tau_{(i)}(Y)$.

Finally, consider a nonisomorphism $g : Y_\lambda \longrightarrow Y_\lambda$, with $ly(\lambda) = i$. Again, we have that $\operatorname{LL}(Y_\lambda) = m - i + 1$, but $\operatorname{LL}(\operatorname{Im} g) < m - i + 1$. Hence $\operatorname{Im} g \subseteq \tau_{(i)}(Y)$, so assumption (C6) holds.

By Theorem 5.2.8, the algebra $\Gamma = \operatorname{End}_A(\tilde{X})^{op}$ obtained in this case (where $\tilde{X} = \bigoplus_{i=1}^{m} \operatorname{Soc}_i X_1$), is left strongly quasihereditary, and $\operatorname{gl.dim} \Gamma \leq m = \operatorname{LL}(X_1)$.

If A is an Artin C-algebra and if X_1 is the direct sum of a complete set of injective indecomposable A-modules, then, using the notation in Section 5.2, the corresponding algebra $\Gamma = \operatorname{End}_A(\tilde{X})^{op}$ is Morita equivalent to $S_A \cong (R_{A^{op}})^{op}$, where $R_{A^{op}}$ is the ADR algebra of A^{op} (see Subsection 3.4.1).

Remark 5.3.1. Let $X_1 \in \operatorname{mod} A$ be a direct sum of modules with simple top. According to [47], the algebra $\operatorname{End}_A(\bigoplus_{i:\, i \geq 1} X_1/\operatorname{Rad}^i X_1)^{op}$ is quasihereditary. By dualising this result, we deduce that the algebra $\operatorname{End}_A(\bigoplus_{i:\, i\geq 1} \operatorname{Soc}_i X_1)^{op}$ is quasihereditary when the module X_1 is a direct sum of modules with simple socle. We have just shown that the algebra $\operatorname{End}_A\left(\bigoplus_{i:\, i\geq 1} \operatorname{Soc}_i X_1\right)^{op}$ is in fact left strongly quasihereditary if no nonisomorphic indecomposable summands of X_1 have the same simple socle.

5.3.2 Iyama's construction

Recall that the representation dimension of an Artin algebra A, $\operatorname{rep.dim} A$, is the minimal possible global dimension of the endomorphism algebra of an A-module which is both a generator and a cogenerator of $\operatorname{mod} A$ (see Subsection 4.2.4).

In [39], Iyama showed that every Artin algebra A has finite representation dimension, thus answering a question with more than 30 years, posed by Auslander. Indeed, Iyama established a much stronger result – he proved that every module $X_1 \in \operatorname{mod} A$ has a complement X' such that $\Gamma = \operatorname{End}_A (X_1 \oplus X')^{op}$ is a quasihereditary algebra. By taking X_1 to be a generator and cogenerator of $\operatorname{mod} A$, the corresponding algebra Γ is quasihereditary, so Γ has finite global dimension and $\operatorname{rep.dim} A < \infty$. In [51], Ringel observed that the algebras Γ constructed by Iyama are in fact left strongly quasihereditary.

In this subsection we show that Iyama's endomorphism algebra Γ fits into the setup described in Section 5.2. Consider the functors $\operatorname{Rad}_A(-,-)$ and $\tau_{\operatorname{Rad}_A(M,-)}$ defined in Appendix B. For $M, N \in \operatorname{mod} A$, we have that

$$\tau_{\operatorname{Rad}_A(M,-)}(N) = \sum_{f\colon f\in \operatorname{Rad}_A(M,N)} \operatorname{Im} f.$$

The underlying algebra A does not need to be an Artin algebra, i.e. we shall assume that A is an arbitrary C-algebra.

Following Iyama's construction in [39] (see also [51]), pick a module X_1 in $\operatorname{mod} A$ and consider the preradical $\tau_{(1)} := \tau_{\operatorname{Rad}_A(X_1,-)}$. For $X_i \neq 0$, define recursively $X_{i+1} := \tau_{(i)}(X_i)$, where $\tau_{(i)} := \tau_{\operatorname{Rad}_A(X_i,-)}$. So

$$X_{i+1} = \tau_{\operatorname{Rad}_A(X_i,-)}(X_i) = \sum_{f\colon f\in \operatorname{Rad}_A(X_i,X_i)} \operatorname{Im} f. \tag{5.4}$$

Note that X_{i+1} is a proper submodule of X_i, since $X_{i+1} = \operatorname{Rad}(\operatorname{End}_A(X_i))X_i$ (that is, X_{i+1} is the radical of X_i as an $\operatorname{End}_A(X_i)$-module). This produces a chain of proper A-submodules, which has to terminate because X_1 has finite length. Let m be largest integer such that $X_m \neq 0$. Note that we are in the setup described in conditions (C1), (C2) and (C3), and in this particular situation $\tilde{X}$ has m layers.

Next, we check that assumptions (C4), (C5) and (C6) are satisfied. This will follow [51, §2] quite closely.

To see that assumption (C4) is satisfied let Y be an indecomposable summand of X_i which is not a summand of X_{i+1}, and let Y' be an indecomposable summand of X_j, for some $j > i$. Consider a map $g : Y' \longrightarrow Y$. By (5.4), $X_{k+1} = \tau_{(k)}(X_k)$ is generated by X_k for every k, $1 \leq k \leq m$ (see also Lemma B.4, part 3). So we have maps

$$X_i{}^{k_1} \xrightarrow{\pi_1} X_{i+1}{}^{k_2} \xrightarrow{\pi_2} Y' \xrightarrow{g} Y \quad .$$

Note that $g \circ \pi_2 \circ \pi_1 \in \operatorname{Rad}_A(X_i{}^{k_1}, Y)$, otherwise the map

$$Y'' \hookrightarrow X_i{}^{k_1} \xrightarrow{\pi_1} X_{i+1}{}^{k_2} \xrightarrow{\pi_2} Y' \xrightarrow{g} Y$$

would be an isomorphism for some indecomposable summand Y'' of X_i, and consequently Y would be a summand of X_{i+1} – a contradiction. So we have that

$$\operatorname{Im} g = \operatorname{Im}(g \circ \pi_2 \circ \pi_1) \subseteq \tau_{\operatorname{Rad}_A(X_i{}^{k_1},-)}(Y) = \tau_{\operatorname{Rad}_A(X_i,-)}(Y) = \tau_{(i)}(Y)$$

(for the penultimate equality see part 5 of Lemma B.4).

We now check (simultaneously) that assumptions (C5) and (C6) are satisfied. We use the notation introduced in Section 5.2. Consider a nonisomorphism $g : Y_\lambda \longrightarrow Y_\nu$, with $ly(\lambda) = ly(\nu) = i$. Note that Y_λ is a summand of X_i, so we have a map

$$X_i \xrightarrow{\pi} Y_\lambda \xrightarrow{g} Y_\nu \quad .$$

The map $g \circ \pi$ belongs to $\operatorname{Rad}_A(X_i, Y_\nu)$, otherwise g would have to be an isomorphism, which is a contradiction. Hence

$$\operatorname{Im} g = \operatorname{Im}(g \circ \pi) \subseteq \tau_{\operatorname{Rad}_A(X_i,-)}(Y_\nu) = \tau_{(i)}(Y_\nu)\,.$$

By Theorem 5.2.8, the corresponding endomorphism algebra $\Gamma = \operatorname{End}_A(\tilde{X})^{op}$, where $\tilde{X} = \bigoplus_{i=1}^{m} X_i$, is left strongly quasihereditary and $\operatorname{gl.dim} \Gamma \leq m$. This gives another proof of Iyama's theorem in [39], which basically mimics Ringel's strategy in [51].

Remark 5.3.2. Suppose A is an Artin C-algebra of finite type. In this case, we may choose the module X_1 in this construction to be the direct sum of all indecomposable A-modules (up to isomorphism). As observed in [51], the left strongly quasihereditary algebra $\Gamma = \operatorname{End}_A(\tilde{X})^{op}$ produced in this case is Morita equivalent to $\operatorname{End}_A(X_1)^{op}$, the Auslander algebra of A.

5.3.3 Cluster-tilted algebras associated with reduced words in Coxeter groups

Geiß–Leclerc–Schröer ([32], [31]) and Iyama–Reiten [40] have shown there is a cluster-tilting category $\underline{\mathcal{C}_\omega}$ associated to every preprojective algebra A and every element ω in the Coxeter group of A. Moreover, to each reduced expression of the element ω there is a corresponding cluster-tilted algebra Γ. As proved in [40] and [32], the algebra Γ is actually a strongly quasihereditary algebra. We show that this class of algebras fits into the construction described in Section 5.2. First, we need to introduce some notation.

Let $Q = (Q_0, Q_1)$ be a finite connected quiver without oriented cycles. As usual, $Q_0 = \{1, \ldots, n\}$ denotes the set of vertices and Q_1 is the set of arrows. Following Gel'fand and Ponomarev ([33]), let

$$A := K\overline{Q}/\langle c \rangle$$

be the associated *preprojective algebra*. Here K is an algebraically closed field, $K\overline{Q}$ is the path algebra of the *double quiver* of Q (which is obtained by adding to each arrow α in Q_1 an arrow α^* pointing in the opposite direction), and c is the element

$$c = \sum_{\alpha \in Q_1} (\alpha^* \alpha - \alpha \alpha^*) \, .$$

Let W_Q be the Coxeter group of Q (see [32, Section 4.1], [43]). For example, if Q is of type A_n, then W_Q is the symmetric group Σ_{n+1}. In the general case W_Q has special generators $s_1, \ldots, s_n$, where s_i is associated to the vertex i of Q. Given $\omega \in W_Q$ we say that $\mathbf{j} = (j_m, \ldots, j_2, j_1)$, $1 \leq j_i \leq n$, is a *reduced expression* for ω if $\omega = s_{j_m} \cdots s_{j_2} s_{j_1}$ and the integer m is minimal (in the sense that, if $\omega = s_{k_l} \cdots s_{k_2} s_{k_1}$, $1 \leq k_i \leq n$, then $l \geq m$).

Let L_j be the simple A-module associated with the vertex j of $\overline{Q}$ and denote its injective hull by Q_j, $j = 1, \ldots, n$. The modules Q_j do not usually lie in $\operatorname{mod} A$, i.e. they may be infinite-dimensional over K.

Fix a reduced expression $\mathbf{j} = (j_m, \ldots, j_2, j_1)$ of some element $\omega \in W_Q$. We may associate to $\mathbf{j}$ a sequence of (hereditary) preradicals $\tau_{(i)}$ in $\operatorname{Mod} A$, constructed in the following way. Define $\delta_{(i)} := \operatorname{Tr}(L_{j_i}, -)$, $1 \leq i \leq m$. Note that for X in $\operatorname{Mod} A$, $\delta_{(i)}(X)$ is largest semisimple submodule of X whose summands are all isomorphic to L_{j_i}. It is easy to check that $\delta_{(i)}$ is a hereditary preradical in $\operatorname{Mod} A$ (recall Definition 1.3.7). For $0 \leq i < m$, set $\tau_{(i)} := \delta_{(1)} \bullet \delta_{(2)} \bullet \cdots \bullet \delta_{(m-i)}$ and $\tau_{(m)} := 0$ (remember the operation $\bullet$ defined in Subsection 1.3.2). Note that $\tau_{(i)}$ is a hereditary preradical in $\operatorname{Mod} A$ as each $\delta_{(i)}$ is hereditary (see Subsection 1.3.2).

Following [32, §2.4], define $X_1 := \tau_{(0)}(\bigoplus_{j=1}^n Q_j) = \bigoplus_{j=1}^n \tau_{(0)}(Q_j)$. The A-module X_1 is finitely generated over C – each socle layer of Q_j is composed of a semisimple module in $\operatorname{mod} A$ as the quiver $\overline{Q}$ has a finite number of arrows. Moreover, X_1 is a direct sum of modules with pairwise distinct simple socle. Note that $\tau_{(0)}(Q_j) = 0$ if and only if $j \neq j_i$ for every $i \in \{1, \ldots, n\}$. The modules X_1 and $X_{i+1} := \tau_{(i)}(X_i)$, $1 \leq i \leq m$, are in the setup of conditions (C1) and (C2) in Section 5.2.

In order to show that the conditions (C3) to (C6) are satisfied, two preliminary lemmas are needed.

Lemma 5.3.3. *For $1 \leq i \leq m$, we have that $\tau_{(i-1)} = \tau_{(i)} \bullet \delta_{(m-i+1)}$ and $X_{i+1} = \tau_{(i)}(\bigoplus_{j=1}^n Q_j)$.*

Proof. The first claim is clear. We prove that $X_{i+1} = \tau_{(i)}(\bigoplus_{j=1}^n Q_j)$ for $1 \leq i \leq m$. Note that identity holds for $i = 0$ by the definition of X_1. Moreover, since $\tau_{(i-1)} = \tau_{(i)} \bullet \delta_{(m-i+1)}$, then $\tau_{(i)} \leq \tau_{(i-1)}$, using the notation in Subsection 1.3.2. So, for $1 \leq i \leq m$, we have

$$\begin{aligned} X_{i+1} = \tau_{(i)}(X_i) &= X_i \cap \tau_{(i)} \left(\bigoplus_{j=1}^n Q_j \right) \\ &= \tau_{(i-1)} \left(\bigoplus_{j=1}^n Q_j \right) \cap \tau_{(i)} \left(\bigoplus_{j=1}^n Q_j \right) = \tau_{(i)} \left(\bigoplus_{j=1}^n Q_j \right). \end{aligned}$$

The second equality is due to the fact that $\tau_{(i)}$ is a hereditary preradical, the third equality uses induction, and the last equality uses that $\tau_{(i)}$ is a subfunctor of $\tau_{(i-1)}$. □

Lemma 5.3.4. *For $1 \leq i \leq m$, the module X_{i+1} has exactly the same decomposition as X_i as a direct sum of indecomposable modules, except that the unique summand of X_i with simple socle $L_{j_{m-i+1}}$, $Y = \tau_{(i-1)}(Q_{j_{m-i+1}})$, is replaced by $\tau_{(i)}(Y) = \tau_{(i)}(Q_{j_{m-i+1}})$ in X_{i+1}. The module X_{i+1} is properly contained in X_i.*

Proof. Consider i, $1 \leq i \leq m$. By Lemma 5.3.3, we have that

$$\begin{aligned} X_i = \tau_{(i-1)} \left(\bigoplus_{j=1}^n Q_j \right) &= \bigoplus_{j=1}^n \tau_{(i-1)}(Q_j) \\ &= \left(\bigoplus_{\substack{j=1, \\ j \neq j_{m-i+1}}}^n \tau_{(i)} \bullet \delta_{(m-i+1)}(Q_j) \right) \oplus \tau_{(i-1)}(Q_{j_{m-i+1}}). \end{aligned}$$

For $j \neq j_{m-i+1}$, we have $\delta_{(m-i+1)}(Q_j) = 0$, so $\tau_{(i)} \bullet \delta_{(m-i+1)}(Q_j) = \tau_{(i)}(Q_j)$, by the definition of the operation $\bullet$. Therefore,

$$X_i = \left(\bigoplus_{\substack{j=1, \\ j \neq j_{m-i+1}}}^n \tau_{(i)}(Q_j) \right) \oplus \tau_{(i-1)}(Q_{j_{m-i+1}}).$$

By Lemma 5.3.3, $X_{i+1} = \bigoplus_{j=1}^{n} \tau_{(i)}(Q_j)$. This proves the first claim.

In order to show that X_{i+1} is a proper submodule of X_i, one needs to check that the module $\tau_{(i)}\left(Q_{j_{m-i+1}}\right)$ is a proper submodule of $\tau_{(i-1)}\left(Q_{j_{m-i+1}}\right)$. For this we need to use that $\mathbf{j} = (j_m, \ldots, j_2, j_1)$ is a reduced word in W_Q. It should be possible to prove that $\tau_{(i)}\left(Q_{j_{m-i+1}}\right) \neq \tau_{(i-1)}\left(Q_{j_{m-i+1}}\right)$ directly, using the Coxeter presentation of W_Q, bearing in mind how this presentation connects to the morphology of the the quiver Q, and hence to the structure of the modules $Q_j \in \operatorname{Mod} A$. This is proved in Section III.1 of [12] (see Proposition III.1.11 – in here the authors use a dual construction, i.e. instead of taking submodules of the injective indecomposable modules Q_j, they take factor modules of the corresponding projective indecomposable modules). □

From Lemma 5.3.4 we conclude that A satisfies assumption (C3). This result also implies that, for each i, $1 \leq i \leq m$, there is a unique indecomposable summand of X_i which is not a summand of X_{i+1}, namely the module $\tau_{(i-1)}\left(Q_{j_{m-i+1}}\right)$. In other words, there is a unique indecomposable module in each layer of $\tilde{X} = \bigoplus_{i=1}^{m} X_i$.

To show that (C4) holds, consider a map $g : Y' \longrightarrow Y$, with Y an indecomposable summand of X_i which is not a summand of X_{i+1} and Y' an indecomposable module in $\operatorname{add} \tilde{X}_{>i}$. Note that

$$Y' = Y' \cap X_{i+1} = Y' \cap \tau_{(i)}(X_i) = \tau_{(i)}(Y').$$

Here the first identity is due to the fact that Y' is contained in X_{i+1}. For the last equality note that $Y' \subseteq X_i$ and that $\tau_{(i)}$ is a hereditary preradical. Because $\tau_{(i)}$ is a preradical, then the image of the map $g|_{\tau_{(i)}(Y')} = g$ is contained in $\tau_{(i)}(Y)$. That is, $\operatorname{Im} g \subseteq \tau_{(i)}(Y)$.

Assumption (C5) is trivially satisfied since there is just one indecomposable module in each layer of $\tilde{X}$.

To check that assumption (C6) holds, consider a nonisomorphism $g : Y \longrightarrow Y$, with $ly(Y) = i$. We must have $Y = \tau_{(i-1)}\left(Q_{j_{m-i+1}}\right)$. In particular, $\operatorname{Soc} Y = L_{j_{m-i+1}}$ and $\tau_{(i-1)}(Y) = Y$ (as $\tau_{(i-1)}$ is idempotent – see Lemma 1.3.9 and Definition 1.3.4). Because g is a nonisomorphism, there is an epic

$$Y/L_{j_{m-i+1}} \overset{h}{\twoheadrightarrow} \operatorname{Im} g.$$

Note that

$$\begin{aligned} Y/L_{j_{m-i+1}} &= Y/\delta_{(m-i+1)}(Y) = \tau_{(i-1)}(Y)/\delta_{(m-i+1)}(Y) = \tau_{(i)} \bullet \delta_{(m-i+1)}(Y)/\delta_{(m-i+1)}(Y) \\ &= \tau_{(i)}\left(Y/L_{j_{m-i+1}}\right), \end{aligned}$$

where the last identity is due to the definition of $\bullet$. This implies that the image of $h|_{\tau_{(i)}(Y/L_{j_{m-i+1}})} = h$ is contained in $\tau_{(i)}(\operatorname{Im} g)$. That is we must have $\tau_{(i)}(\operatorname{Im} g) = \operatorname{Im} g$. As a consequence we get that $\operatorname{Im} g \subseteq \tau_{(i)}(Y)$.

This proves that the left strongly quasihereditary algebra investigated in [32], [31] and also in [40] fits into the construction outlined in Section 5.2. By applying Theorem 5.2.8 to the setup described in this subsection, we get a new proof of the fact that algebra $\Gamma = \operatorname{End}_A(\tilde{X})^{op}$, $\tilde{X} = \bigoplus_{i=1}^{m} X_i$, is left strongly quasihereditary. Note that the global dimension of Γ is not greater than the length m of the chosen reduced word in W_Q. As we shall see in Section 5.4, the algebra Γ obtained in this case is actually a LUSQ algebra (recall the definition of LUSQ algebra in Section 4.3).

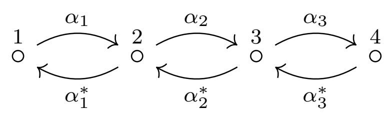
As investigated in [32], [31], [40], [41], the algebra Γ can be realised as a cluster-tilted algebra. Define $\mathcal{C}_\omega$ to be the category of quotients of modules in $\operatorname{add}\tilde{X}$ – it can be proved that $\mathcal{C}_\omega$ does not depend on the choice of a reduced expression for ω. It turns out that the category $\mathcal{C}_\omega$ is a Frobenius subcategory of $\operatorname{mod} A$ (see [35]) and the $\mathcal{C}_\omega$-projective-injective modules are the indecomposable summands of X_1. The corresponding stable category $\underline{\mathcal{C}_\omega}$ is a triangulated 2-Calabi-Yau category and $\operatorname{add}\tilde{X}$ embedds in $\underline{\mathcal{C}_\omega}$ as a 2-cluster-tilting subcategory in the sense of Iyama and Yoshino (see [32], [31], [40], and [41]).

Remark 5.3.5. Note that we barely used any data specific to the preprojective algebra when checking that Γ fits into the setup of Section 5.2. In fact, we can reproduce this particular construction in a general setting.

Consider a C-algebra A and let $L_1, \ldots, L_n$ be pairwise nonisomorphic simple modules. Let $X_0 = \bigoplus_{j=1}^{n} \widehat{Q}_j$, where each $\widehat{Q}_j$ is in $\operatorname{Mod} A$ and has simple socle L_j. Consider a sequence $\mathbf{j} = (j_m, \ldots, j_2, j_1)$, with $1 \leq j_i \leq n$. Then we may define the chain of hereditary preradicals $\tau_{(m)} \leq \cdots \leq \tau_{(1)} \leq \tau_{(0)}$ as before. If $X_1 := \tau_{(0)}(X_0)$ is in $\operatorname{mod} A$ and the inclusion of $\tau_{(i)}(\widehat{Q}_{j_{m-i+1}})$ in $\tau_{(i-1)}(\widehat{Q}_{j_{m-i+1}})$ is proper for every $0 \leq i \leq m$ then assumptions (C1) to (C6) are satisfied (and the arguments are exactly as the ones used previously). By Theorem 5.2.8, the resulting algebra $\Gamma = \operatorname{End}_A(\tilde{X})^{op}$ is then left strongly quasihereditary with $\operatorname{gl.dim}\Gamma \leq m$.

Next, we compute an elementary example of a cluster-tilted algebra Γ, with the purpose of illustrating the previous construction and the statement of Theorem 5.2.8.

Example 5.3.6. Let A be the preprojective algebra of type A_4. So A is the quiver algebra of

$\overline{Q} =$ 

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bound by the relations $\alpha_1^*\alpha_1 = 0$, $\alpha_1\alpha_1^* = \alpha_2^*\alpha_2$, $\alpha_2\alpha_2^* = \alpha_3^*\alpha_3$ and $\alpha_3\alpha_3^* = 0$.

The injective A-modules (which are in fact projective-injective modules) are given by

$$Q_1 = \begin{array}{c} 4 \\ | \\ 3 \\ | \\ 2 \\ | \\ 1 \end{array}, \quad Q_2 = \begin{array}{ccccc} & & 3 & & \\ & / & & \backslash & \\ & 2 & & & 4 \\ / & & \backslash & / & \\ 1 & & 3 & & \\ \backslash & / & & & \\ & 2 & & & \end{array}, \quad Q_3 = \begin{array}{ccccc} & 2 & & & \\ / & & \backslash & & \\ 1 & & 3 & & \\ \backslash & / & & \backslash & \\ & 2 & & & 4 \\ & & \backslash & / & \\ & & & 3 & \end{array}, \quad Q_4 = \begin{array}{c} 1 \\ | \\ 2 \\ | \\ 3 \\ | \\ 4 \end{array}.$$

Consider the reduced word $\mathbf{j} = (2, 3, 1)$. Using the previous notation, we have $m = 3$, $j_1 = 1$, $j_2 = 3$ and $j_3 = 2$. Consequently, $\delta_{(1)} = \mathrm{Tr}\,(L_1, -)$, $\delta_{(2)} = \mathrm{Tr}\,(L_3, -)$ and $\delta_{(3)} = \mathrm{Tr}\,(L_2, -)$. Set $X_0 = \bigoplus_{j=1}^4 Q_j$.

We have that $\tau_{(2)}(X_0) = \delta_{(1)}(X_0) = L_1$. So $X_3 = L_1$ (see Lemma 5.3.3).

Note that $\delta_{(2)}(X_0) = L_3$ and $\delta_{(1)}(X_0/\delta_{(2)}(X_0)) = L_1$. So $\delta_{(1)} \bullet \delta_{(2)}(X_0) = L_1 \oplus L_3$. That is $\tau_{(1)}(X_0) = L_1 \oplus L_3 = X_2$ (see Lemma 5.3.3).

Finally, $\delta_{(3)}(X_0) = L_2$ and $\delta_{(2)}(X_0/\delta_{(3)}(X_0)) = L_3 \oplus L_3$. Thus

$$\delta_{(2)} \bullet \delta_{(3)}(X_0) = \begin{array}{c} 3 \\ | \\ 2 \end{array} \oplus L_3.$$

Now $\delta_{(1)}(X_0/\delta_{(2)} \bullet \delta_{(3)}(X_0)) = L_1 \oplus L_1$, hence

$$\delta_{(1)} \bullet \delta_{(2)} \bullet \delta_{(3)}(X_0) = L_1 \oplus \begin{array}{ccc} 1 & & 3 \\ \backslash & & / \\ & 2 & \end{array} \oplus L_3 = \tau_{(0)}(X_0) = X_1.$$

The module $\tilde{X} = X_1 \oplus X_2 \oplus X_3$ has 3 indecomposable summands, up to isomorphism. The summand

$$\begin{array}{ccc} 1 & & 3 \\ \backslash & & / \\ & 2 & \end{array}$$

is in layer 1 of $\tilde{X}$, and we shall denote it by $Y_{1,1}$. The summand L_3 is in layer 2 of $\tilde{X}$, and we will denote it by $Y_{2,1}$. The indecomposable module L_1 is in layer 3 of $\tilde{X}$, and it shall be represented by $Y_{3,1}$.

Set $\Psi = \{(1, 1), (2, 1), (3, 1)\}$. According to Theorem 5.2.8, the algebra $\Gamma = \mathrm{End}_A(\tilde{X})^{op}$ is (left strongly) quasihereditary with respect to the poset $(1, 1) \prec (2, 1) \prec (3, 1)$, where the label $(i, 1)$ corresponds to the projective indecomposable Γ-module $P_{i,1} = \mathrm{Hom}_A(\tilde{X}, Y_{i,1})$.

The algebra Γ is Morita equivalent to its basic version. That is, Γ is Morita equivalent to $\Gamma' = \mathrm{End}_A(Y_{(1,1)} \oplus Y_{(2,1)} \oplus Y_{(3,1)})^{op}$, and the algebra Γ' is isomorphic to the path algebra of the quiver

$$\overset{(3,1)}{\circ} \longrightarrow \overset{(1,1)}{\circ} \longleftarrow \overset{(2,1)}{\circ} \quad .$$

The projective indecomposable Γ'-modules can be represented as

$$P_{1,1} = L_{1,1} = \Delta(1,1), \;\; P_{2,1} = \begin{array}{c} (2,1) \\ \mid \\ (1,1) \end{array} = \Delta(2,1), \;\; P_{2,1} = \begin{array}{c} (3,1) \\ \mid \\ (1,1) \end{array} = \Delta(3,1).$$

Note that every standard module is projective, so Γ' is a left strongly quasihereditary algebra with respect to $(\Psi, \preceq)$ (see Definition 4.3.1), as predicted by Theorem 5.2.8.

The injective indecomposable Γ'-modules are given by

$$Q_{1,1} = \begin{array}{ccccc} (3,1) & & & & (2,1) \\ & \searrow & & \swarrow & \\ & & (1,1) & & \end{array}, \;\; Q_{2,1} = L_{2,1}, \;\; Q_{3,1} = L_{3,1}.$$

Observe that all the costandard modules are simple, i.e. $\nabla(i,1) = L_{i,1}$. As a consequence, the projective modules $P_{i,1}$ lie in $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$, which proves that Γ' is in fact a LUSQ algebra with respect to $(\Psi, \preceq)$ (see Definition 5.4.1). This is hinting at a general phenomenon which will be discussed in the next section.

5.4 Ultra strongly quasihereditary endomorphism algebras

In this section we provide sufficient conditions for which the endomorphism algebras $\Gamma = \mathrm{End}_A(\bigoplus_{i=1}^m X_i)^{op}$ (obtained by the method described in Section 5.2) are LUSQ algebras. The slogan for this section should be the following: if the initial module X_1 "behaves like" an injective module, then the corresponding (left strongly quasihereditary) algebra $(\Gamma, \Psi, \preceq)$ is a LUSQ algebra. For ease of reference, we restate the definition of LUSQ algebra, which is dual to that of a RUSQ algebra (see Subsection 2.5.1).

Definition 5.4.1. Let $(B, \Phi, \sqsubseteq)$ be a left strongly quasihereditary algebra (as in Definition 4.3.1). The algebra $(B, \Phi, \sqsubseteq)$ is a *left ultra strongly quasihereditary algebra* (LUSQ algebra, for short) if, for every $i \in \Phi$, P_i belongs to $\mathcal{F}(\nabla)$ whenever $\nabla(i)$ is a simple module.

This property is preserved under Morita equivalence.

5.4.1 Setup

Using the notation introduced in Section 5.2, let X_1 be a module in $\operatorname{mod} A$ such that any pair of nonisomorphic indecomposable summands of X_1 have nonisomorphic simple socle. Without loss of generality we may suppose that X_1 is basic, that is, we can assume that $X_1 = \bigoplus_{i=1}^{n} Y_{i,1}$ where each $Y_{i,1}$ has simple socle and $\operatorname{Soc} Y_{i,1} \ncong \operatorname{Soc} Y_{k,1}$ for $i \neq k$. Suppose further that there are preradicals $\tau_{(1)}, \ldots, \tau_{(m)}$, so that assumptions (C1) to (C6) hold for this set of data. In this special situation, the indecomposable summands of the module $\tilde{X}$ are isomorphic to particular submodules of the modules $Y_{i,1}$ because the modules $Y_{i,1}$ have simple socle. That is, all the summands of $\tilde{X}$ are of the form $Y_{i,j}$, $1 \leq i \leq n$, $1 \leq j \leq l_i$ (for some $l_i \in \mathbb{Z}_{>0}$), so that for each i there is a filtration

$$0 \subset Y_{i,l_i} \subset \cdots \subset Y_{i,2} \subset Y_{i,1},$$

and the exact sequences (5.2) are of the form

$$0 \longrightarrow P_{i,j+1} \xrightarrow{\iota_*} P_{i,j} \longrightarrow \Delta(i,j) \longrightarrow 0 \quad ,$$

with $P_{i,j} = \operatorname{Hom}_A(\tilde{X}, Y_{i,j})$ (formally, $Y_{i,j+1} = \tau_{(ly(i,j))}(Y_{i,j})$).

Throughout the rest of this chapter we shall assume that the module X_1 is as described above (i.e. $X_1 = \bigoplus_{i=1}^{n} Y_{i,1}$, each $Y_{i,1}$ has simple socle and $\operatorname{Soc} Y_{i,1} \ncong \operatorname{Soc} Y_{k,1}$ for $i \neq k$), and $(\Gamma, \Psi, \preceq)$ will always denote a left strongly quasihereditary algebra which is obtained under these circumstances. We may assume that

$$\Psi = \{(i,j) : 1 \leq i \leq n,\ 1 \leq j \leq l_i\}.$$

In this special case, the projective indecomposable Γ-modules have a unique Δ-filtration,

$$0 \subset P_{i,l_i} \subset \cdots \subset P_{i,2} \subset P_{i,1},$$

and the algebra Γ is a WLUSQ algebra, as investigated in Section 4.3.

5.4.2 Results

It is useful to call to mind the definition of a WLUSQ algebra (see Subsection 4.3.1), and the basic results about these algebras deduced in Subsection 4.3.2.

As seen in Example 4.3.10, a WLUSQ algebra $(B, \Phi, \sqsubseteq)$ does not necessarily satisfy the condition

($\diamondsuit$) $\nabla(i,j)/L_{i,j} \cong \nabla(i,j-1)$, for all $(i,j) \in \Phi$ (where $\nabla(i,0) := 0$).

In particular, WLUSQ algebras are not usually LUSQ, and the same conclusions apply to the (WLUSQ) algebras $(\Gamma, \Psi, \preceq)$ just described.

The following straightforward lemma will be used later on.

Lemma 5.4.2. *Let $(B, \Phi, \sqsubseteq)$ be a WLUSQ algebra, and suppose that B satisfies property $(\diamondsuit)$. Then the costandard B-modules are uniserial, and $\nabla(i, j)$ has composition factors $L_{i,j}, L_{i,j-1} \ldots, L_{i,1}$, ordered from the socle to the top. Given M in $\mathcal{F}(\nabla)$, the number of costandard modules appearing in a ∇-filtration of M is given by $\sum_{i=1}^{n}[M : L_{i,1}]$, and $\operatorname{Top} M$ is a direct sum of modules of type $L_{x,1}$.*

Proof. Only the last assertion needs to be justified. Let M be in $\mathcal{F}(\nabla)$. Since $\mathcal{F}(\nabla)$ is closed under factor modules (see Lemma 4.3.4), then $\operatorname{Top} M$ is a ∇-good module. This implies that $\operatorname{Top} M$ is a direct sum of modules of type $L_{x,1}$. □

The next result gives sufficient conditions for the property $(\diamondsuit)$ to hold for the WLUSQ algebras $(\Gamma, \Psi, \preceq)$ introduced in Subsection 5.4.1.

Proposition 5.4.3. *Using the previous notation (as in Subsection 5.4.1), suppose that, for $1 \leq i \leq n$, every monic $f : Y_{i,l_i} \longrightarrow Y_{i,1}$ factors through the inclusion ι of Y_{i,l_i} in $Y_{i,1}$ as $f = f' \circ \iota$ for some map f'. Then the WLUSQ algebra $(\Gamma, \Psi, \preceq)$ satisfies property $(\diamondsuit)$. In particular, the algebra $(\Gamma, \Psi, \preceq)$ satisfies property $(\diamondsuit)$ if one of the following conditions is satisfied:*

1. *the ring of scalars is an algebraically closed field, i.e. A is a K-algebra and K is an algebraically closed field;*
2. *the modules $Y_{i,1}$, $1 \leq i \leq n$, are injective;*
3. *each module $Y_{i,1}$, $1 \leq i \leq n$, is a characteristic submodule*[1] *of its injective hull.*

Proof. Consider the inclusion $\iota : Y_{i,l_i} \longrightarrow Y_{i,1}$ and apply the functor $\operatorname{Hom}_A(-, \tilde{X})$ to this morphism. We get the map

$$\begin{array}{ccc} \operatorname{Hom}_A(Y_{i,1}, \tilde{X}) & \xrightarrow{\quad \iota_* \quad} & \operatorname{Hom}_A(Y_{i,l_i}, \tilde{X}) \\ & \searrow\!\!\!\searrow \qquad \nearrow & \\ & \operatorname{Im} \iota_* & \end{array} \quad . \tag{5.5}$$

[1]See Subsection 1.3.1 for the definition of characteristic submodule.

Note that $\operatorname{Im}\iota_*$ is a Γ^{op}-submodule of $\operatorname{Hom}_A(Y_{i,l_i}, \tilde{X})$. Write $P'_{i,j} := \operatorname{Hom}_A(Y_{i,j}, \tilde{X})$. The modules $P'_{i,j}$, $(i,j) \in \Psi$, form a complete list of projective indecomposable Γ^{op}-modules. Write $L'_{i,j} := \operatorname{Top} P'_{i,j}$. By the diagram (5.5), $\operatorname{Im}\iota_*$ is a Γ^{op}-submodule of P'_{i,l_i} with simple top $L'_{i,1}$.

We claim that the composition factor $L'_{i,1}$ appears exactly once in the composition series of P'_{i,l_i}, namely in the top of $\operatorname{Im}\iota_*$. For this consider a nonzero map $f_* : P'_{i,1} \longrightarrow P'_{i,l_i}$. We have that $f_* = \operatorname{Hom}_A(f, \tilde{X})$ for some $f : Y_{i,l_i} \longrightarrow Y_{i,1}$ in $\operatorname{mod} A$. Proposition 5.2.2 implies that the inclusion morphism $\iota : Y_{i,l_i} \longrightarrow Y_{i,1}$ is a right (minimal) add $\tilde{X}_{>k}$-approximation of $Y_{i,1}$, where $k := ly(i, l_i - 1)$ (define $ly(i,0) := 0$). To see this, note that ι is obtained by composing the inclusion morphisms $Y_{i,j+1} \longrightarrow Y_{i,j}$, and each of these morphisms is a right minimal add $\tilde{X}_{>ly(i,j)}$-approximation (as $Y_{i,j+1} = \tau_{(ly(i,j))}(Y_{i,j})$). Since ι is a right add $\tilde{X}_{>k}$-approximation of $Y_{i,1}$, then $f = \iota \circ s$, for some map $s : Y_{i,l_i} \longrightarrow Y_{i,l_i}$. If s is not an isomorphism, then condition (C6) implies that $s = 0$ (there is no submodule of Y_{i,l_i} in a layer above $ly(i,l_i)$), and hence $f_* = 0$, which cannot happen. So s has to be an isomorphism, and consequently f is a monic. By the assumption in the statement of the lemma, there is a morphism $f' : Y_{i,1} \longrightarrow Y_{i,1}$ such that $f = f' \circ \iota$. Since all the modules involved in this composition have the same socle, and since the maps f and ι are monic, then f' has to be monic too. As f' is an endomorphism, then f' is a bijection. The identity $f = f' \circ \iota$ implies that $\operatorname{Im} f_* = \operatorname{Im}\iota_*$. Therefore the composition factor $L'_{i,1}$ appears exactly once in the composition series of P'_{i,l_i} (namely in the top of $\operatorname{Im}\iota_*$).

Let D be the standard duality for Γ^{op} (see Subsection 1.2.1.1). We have that $D(P'_{i,l_i}) = Q_{i,l_i}$ (so Q_{i,l_i} is the injective Γ-module with socle L_{i,l_i}), and $[Q_{i,l_i} : L_{i,1}] = 1$ because $[P'_{i,l_i} : L'_{i,1}] = 1$. Note that $\nabla(i,l_i) \subseteq Q_{i,l_i}$, and by Lemma 4.3.6, $[\nabla(i,l_i) : L_{i,1}] \neq 0$. This implies that $[\nabla(i,l_i) : L_{i,1}] = 1$.

Recall that $\mathcal{F}(\nabla)$ is closed under quotients since Γ a left strongly quasihereditary algebra (Theorem 5.2.8). If $l_i = 1$, then $\nabla(i,l_i) = \nabla(i,1) = L_{i,1}$ by Lemma 4.3.6. Suppose now that $l_i > 1$. Note that $\nabla(i,l_i)/L_{i,l_i}$ is a ∇-good module. Since $[\nabla(i,l_i)/L_{i,l_i} : L_{i,1}] = 1$, Lemma 4.3.6 forces $\nabla(i,l_i)/L_{i,l_i}$ to be isomorphic to $\nabla(i,l_i-1)$. By proceeding inductively in this fashion, we see that $\nabla(i,j)/L_{i,j} \cong \nabla(i,j-1)$, where $\nabla(i,0) := 0$. This proves the first part of the proposition.

Next, we show Γ satisfies the lifting property in the statement of the lemma if one of the properties 1, 2 or 3 is satisfied.

First suppose that condition 1 holds, i.e. assume that the underlying ring of scalars is an algebraically closed field. Given a monic $f : Y_{i,l_i} \longrightarrow Y_{i,1}$, we have that $f = \iota \circ s$

for some map $s : Y_{i,l_i} \longrightarrow Y_{i,l_i}$: this is because ι is a right (minimal) add $\tilde{X}_{>k}$-approximation of $Y_{i,1}$, where $k = ly(i, l_i - 1)$ (see the first part of the proof). Condition (C6) implies that $\mathrm{End}_A(Y_{i,l_i})$ is a division algebra. Since K is algebraically closed then $\mathrm{End}_A(Y_{i,l_i}) \cong K$. So the map s is a scalar multiple of $1_{Y_{i,l_i}}$. Thus $f = (k1_{Y_{i,1}}) \circ \iota$, where $s = k1_{Y_{i,l_i}}$ and $k \in K$. Hence the lifting property holds in this case.

Suppose now that condition 2 holds. By the injectivity of $Y_{i,1}$, every morphism $f : Y_{i,l_i} \longrightarrow Y_{i,1}$ factors through the inclusion ι as $f = f' \circ \iota$ for some map f'.

Finally, suppose that condition 3 holds, i.e. assume that each $Y_{i,1}$ is a characteristic submodule of its injective hull, $Q_0(Y_{i,1})$, for $1 \leq i \leq n$. Using the injectivity of $Q_0(Y_{i,1})$, we conclude the following. Given a morphism $f : Y_{i,l_i} \longrightarrow Y_{i,1}$, we have that $\iota' \circ f = f' \circ \iota' \circ \iota$ for some map $f' : Q_0(Y_{i,1}) \longrightarrow Q_0(Y_{i,1})$, where ι denotes the inclusion of Y_{i,l_i} in $Y_{i,1}$, and ι' denotes the inclusion of $Y_{i,1}$ in $Q_0(Y_{i,1})$. Since $Y_{i,1}$ is a characteristic submodule of $Q_0(Y_{i,1})$, then $\mathrm{Im}(f' \circ \iota') \subseteq Y_{i,1}$. But then the morphism f factors through ι as $f = f'' \circ \iota$ for some map f''. □

Remark 5.4.4. Note that the module X_1 in Subsection 5.3.1 satisfies the conditions described in Subsection 5.4.1. Therefore the algebra Γ constructed in Subsection 5.3.1 (using the socle series of X_1) is a WLUSQ algebra. Suppose additionally that the indecomposable summands $Y_{i,1}$ of X_1 are injective (or, more generally, that the summands $Y_{i,1}$ are characteristic submodules of their injective hulls). Then, by Proposition 5.4.3, the WLUSQ algebra Γ constructed in Subsection 5.3.1 satisfies property ($\diamondsuit$).

Observe that the module X_1 used in Subsection 5.3.3 is also in the setup of Subsection 5.4.1. Therefore the corresponding cluster-tilted algebra Γ is a WLUSQ algebra. Proposition 5.4.3 implies that the cluster-tilted algebra Γ satisfies property ($\diamondsuit$), as the underlying ring of scalars is an algebraically closed field[2].

We may also apply Iyama's construction (see Subsection 5.3.2) to a module $X_1 = \bigoplus_{i=1}^{n} Y_{i,1}$ such that each $Y_{i,1}$ has simple socle and $Y_{i,1} \not\cong Y_{k,1}$ for $i \neq k$ (as described in Subsection 5.4.1). If conditions 1, 2 or 3 in Proposition 5.4.3 hold for X_1, then we obtain a WLUSQ algebra Γ which satisfies property ($\diamondsuit$).

We would like to use the general construction outlined in Section 5.2 to produce LUSQ algebras. With this in mind, consider the following condition regarding the summands $Y_{i,j}$ of the initial module X_1:

[2]In this situation the indecomposable summands of X_1 actually satisfy condition 3 in Proposition 5.4.3, which is independent of the field.

(D) every morphism $Y_{i,j} \longrightarrow Y_{k,1}$ factors through the inclusion of $Y_{i,j}$ in $Y_{i,1}$, for every i, k and j ($1 \leq i, k$, $i, k \leq n$ and $1 \leq j \leq l_i$).

The main goal of this section is to prove the following result.

Theorem 5.4.5. *Using the previous notation (as in Subsection 5.4.1), assume that condition (D) holds. Then the corresponding algebra* $(\Gamma, \Psi, \preceq)$ *is a LUSQ algebra. In particular,* Γ *satisfies property* $(\diamondsuit)$ *and the* Γ*-module* $P_{i,1}$ *is isomorphic to* $T(i, l_i)$ *for* $1 \leq i \leq n$.

Observe that the quasihereditary structure of a LUSQ algebra has been described in this thesis. To be precise, we know how the costandard and the tilting modules over a LUSQ algebra look like (see Proposition 4.3.2 and Theorem 4.3.3). Before proving Theorem 5.4.5, we discuss some of its consequences.

Remark 5.4.6. Note that condition (D) is satisfied if, for instance, every $Y_{i,1}$ is an injective (indecomposable) module. So the claim of Theorem 5.4.5 holds in general if the inicial module X_1 is an injective module in $\operatorname{mod} A$.

In particular, Theorem 5.4.5 holds for the algebra Γ constructed in Subsection 5.3.1 if the summands $Y_{i,1}$ of X_1 are injective. By taking A to be an Artin algebra and X_1 to be the sum of all injective indecomposable A-modules, we get that $S_A \cong (R_{A^{op}})^{op}$ is a LUSQ algebra (which had already been deduced in Subsection 3.4.1).

Theorem 5.4.5 also holds for the algebras Γ obtained by applying Iyama's construction (see Subsection 5.3.2) to an injective module X_1 in $\operatorname{mod} A$.

Consider now the cluster-tilted algebra Γ described in Subsection 5.3.3. The lifting property (D) is also satisfied in this case because the summands $Y_{i,1}$ of X_1 are $\mathcal{C}_\omega$-projective-injective objects in the Frobenius category $\mathcal{C}_\omega$, which contains $\operatorname{add} \tilde{X}$ (see Subsection 5.3.3). Consequently, the cluster-tilted algebra Γ is a LUSQ algebra. This result was implicitly proved in [32] – see Theorem 11.1.

In order to prove Theorem 5.4.5 some preparatory results are needed.

Lemma 5.4.7. *Using the previous notation (as in Subsection 5.4.1), suppose that condition (D) is satisfied. Then the WLUSQ algebra* $(\Gamma, \Psi, \preceq)$ *satisfies property* $(\diamondsuit)$. *In particular, the costandard* Γ*-modules are uniserial, and* $\nabla(i, j)$ *has composition factors* $L_{i,j}, L_{i,j-1} \ldots, L_{i,1}$*, ordered from the socle to the top. Moreover, for a module* M *in* $\mathcal{F}(\nabla)$*, the number of costandard modules appearing in a* ∇*-filtration of* M *is given by* $\sum_{i=1}^{n}[M : L_{i,1}]$*, and* $\operatorname{Top} M$ *is a direct sum of modules of type* $L_{x,1}$.

Proof. Since condition (D) is satisfied, Proposition 5.4.3 implies that property $(\diamondsuit)$ holds for Γ. The remaining claims in the statement of this lemma follow from Lemma 5.4.2. $\square$

Recall the notion of Δ-semisimple module, introduced in Section 3.2. In a similar way, a module is said to be ∇-*semisimple* if it is a direct sum of costandard modules. As seen in Section 3.2, the Δ-semisimple modules are specially nice when the underlying algebra is a RUSQ algebra: for instance, the property of being Δ-semisimple is closed under taking submodules in this case (see Corollary 3.2.3). Naturally, the ∇-semisimple modules over a LUSQ algebra satisfy corresponding dual properties. We wish to show that the algebra Γ is a LUSQ algebra when property (D) holds. In order to achieve this, we start by showing that the ∇-semisimple Γ-modules are particularly well behaved when condition (D) is satisfied.

Lemma 5.4.8. *Using the previous notation (as in Subsection 5.4.1), assume that condition (D) is satisfied. Let X in $\operatorname{mod}\Gamma$ and consider a short exact sequence*

$$0 \longrightarrow \nabla(i,j) \longrightarrow X \longrightarrow \nabla(k,l) \longrightarrow 0, \tag{5.6}$$

with $(i,j),(k,l) \in \Psi$. If $\operatorname{Top} X \ncong \operatorname{Top} \nabla(k,l)$ then (5.6) splits.

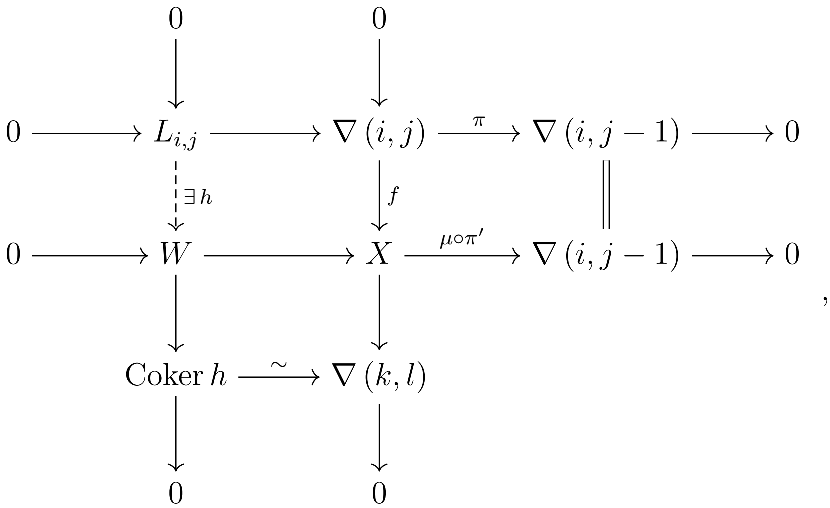
Proof. We prove this statement by induction on j. For $j = 1$, we have that $\nabla(i,j) = \nabla(i,1) = L_{i,1}$. Thus, if $\operatorname{Top} X \ncong \operatorname{Top} \nabla(k,l)$, then the corresponding short exact sequence (5.6) splits.

Suppose now that $j \geq 2$ and consider the pushforward diagram (recall that $\nabla(i,j)/L_{i,j} \cong \nabla(i,j-1)$, by Lemma 5.4.7)

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & L_{i,j} & = & L_{i,j} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \nabla(i,j) & \xrightarrow{f} & X & \longrightarrow & \nabla(k,l) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} & & \| & & \\
0 & \longrightarrow & \nabla(i,j-1) & \xrightarrow{f'} & Z & \longrightarrow & \nabla(k,l) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}, \tag{5.7}$$

where $\operatorname{Top} X \ncong \operatorname{Top} \nabla(k,l)$ (so $\operatorname{Top} X \cong \operatorname{Top} \nabla(k,l) \oplus \operatorname{Top} \nabla(i,j) \cong L_{k,1} \oplus L_{i,1}$). We want to prove that f is a split monic.

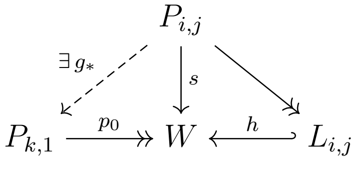
We must have $\operatorname{Top} Z \ncong \operatorname{Top} \nabla(k,l)$. Indeed, $\operatorname{Top} Z \cong \operatorname{Top} \nabla(k,l)$, together with $\operatorname{Top} X \ncong \operatorname{Top} \nabla(k,l)$, implies that $\operatorname{Top} X \cong \operatorname{Top} Z \oplus L_{i,j} \cong L_{k,1} \oplus L_{i,j}$, which leads to a contradiction as $j \geq 2$. So $\operatorname{Top} Z \ncong \operatorname{Top} \nabla(k,l)$ and the short exact sequence in the bottom of (5.7) splits by induction. Let μ be such that $\mu \circ f' = 1_{\nabla(i,j-1)}$, and consider the diagram



, (5.8)

where $W := \operatorname{Ker}(\mu \circ \pi')$.

Our goal is to prove that the central column of (5.8) splits. Suppose, by contradiction, that this exact sequence does not split. Then the left hand column of (5.8) does not split either. As a consequence, the modules W and $\nabla(k,l)$ have the same top, namely $\operatorname{Top} W \cong \operatorname{Top} \nabla(k,l) \cong L_{k,1}$. Therefore, there is a commutative diagram



,

where the map s is the composition of the natural projection with h. We have that $g_* = \operatorname{Hom}_A(\tilde{X}, g)$, for some map $g : Y_{i,j} \longrightarrow Y_{k,1}$ (see Proposition 1.2.4). By condition (D), $g = g' \circ \iota$ for some map $g' : Y_{i,1} \longrightarrow Y_{k,1}$, where $\iota : Y_{i,j} \longrightarrow Y_{i,1}$ is the inclusion map. Write $\iota_* := \operatorname{Hom}_A(\tilde{X}, \iota)$ and ${g'}_* := \operatorname{Hom}_A(\tilde{X}, g')$. Note that $p_0 \circ {g'}_* \neq 0$: the identity $p_0 \circ {g'}_* = 0$ implies that $s = p_0 \circ g_* = p_0 \circ {g'}_* \circ \iota_* = 0$, which is a contradiction.

As $p_0 \circ g'_* : P_{i,1} \longrightarrow W$ is nonzero, then the simple module $L_{i,1}$ must be a composition factor of W. Since $j \geq 2$, then $L_{i,1}$ must be a composition factor of $\operatorname{Coker} h \cong \nabla(k,l)$ (look at the left hand column of (5.8)). So $k = i$.

Observe that the module W has a unique composition factor of the form $L_{x,1}$, namely the module $L_{k,1}$ appearing in its simple top. But then the map g'_* has to be an epic: if g'_* was not epic then the nonzero maps $p_0 \circ g'_*$ and p_0 would give rise to distinct composition factors of W of type $L_{x,1}$. Thus g'_* is an epic, and hence it is an isomorphism (as $k = i$). So g_* is a monic, and we get a diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P_{i,j} & \xrightarrow{g_*} & P_{i,1} & \longrightarrow & \operatorname{Coker} g_* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle p_0} & & \downarrow{\scriptstyle \exists t} & & \\
0 & \longrightarrow & L_{i,j} & \xrightarrow{h} & W & \longrightarrow & \nabla(i,l) & \longrightarrow & 0
\end{array},$$

where t is an epic. By Lemma 4.3.5,

$$P_{i,j} \cong \operatorname{Im} g_* \subseteq \operatorname{Tr}\left(\bigoplus_{(k,l):\,(k,l) \not\preceq (i,j-1)} P_{k,l}, P_{i,1} \right) = P_{i,j},$$

i.e. $\operatorname{Im} g_* = P_{i,j}$ and $\operatorname{Coker} g_* = P_{i,1}/P_{i,j}$. In particular all composition factors of $\operatorname{Coker} g_*$ are of the form $L_{x,y}$, with $(x,y) \preceq (i, j-1)$ (since $P_{i,1}/P_{i,j}$ is filtered by the standard modules $\Delta(i,1), \ldots, \Delta(i,j-1)$). So $(k,l) = (i,l) \preceq (i,j-1) \prec (i,j)$, as $L_{i,l}$ is a composition factor of $\operatorname{Coker} g_*$. By part 2 of Lemma 1.4.5 the central column of (5.8) splits – a contradiction. $\square$

As previously mentioned, we wish to prove Theorem 5.4.5, i.e. we want to show that the WLUSQ algebra $(\Gamma, \Psi, \preceq)$ is a LUSQ algebra whenever condition (D) is satisfied. Observe that the class of ∇-semisimple modules over a LUSQ algebra is closed under factor modules (consider the dual version of Corollary 3.2.3). Next, we prove that the class of ∇-semisimple Γ-modules is closed under quotients when condition (D) is satisfied, which is in accordance with the statement of Theorem 5.4.5.

The following results can be deduced by dualising the reasoning in the proofs of Corollary 3.2.3 and Proposition 3.2.7.

Corollary 5.4.9. *Using the previous notation (as in Subsection 5.4.1), assume that condition (D) is satisfied. Let X be in $\mathcal{F}(\nabla)$. Then X is ∇-semisimple if and only if the number of simple summands of* $\operatorname{Top} X$ *coincides with the number of factors in a ∇-filtration of X. Moreover, any quotient of a ∇-semisimple module is still ∇-semisimple.*

Proof. Let $(\Gamma, \Psi, \preceq)$ be an algebra obtained using the setup in Subsection 5.4.1 (so Γ is a WLUSQ algebra, to be precise). Assume that condition (D) is satisfied. We prove this result by dualising the arguments in the proof of Corollary 3.2.3.

Consider a ∇-good module X. Let $\mathcal{P}(X)$ be the following statement: "the number of simple summands of $\operatorname{Top} X$ coincides with the number of factors in a ∇-filtration of X". By Lemma 5.4.7, $\mathcal{P}(X)$ holds if and only if the composition factors of X of type $L_{x,1}$ are exactly the summands of its top. From this, we see that the truth of $\mathcal{P}(X)$ implies the truth of $\mathcal{P}(X')$ whenever X' a factor module of X (note that every epic $X \longrightarrow X'$ induces an epic $\operatorname{Rad} X \longrightarrow \operatorname{Rad} X'$ – see Example 1.3.8 and Remark 1.3.10).

If X is a ∇-semisimple module then the assertion $\mathcal{P}(X)$ is obviously true. Suppose now that $\mathcal{P}(X)$ holds for $X \in \mathcal{F}(\nabla)$. We want to show that X is ∇-semisimple. We prove this by induction on the number z of factors in a ∇-filtration of X. If $z = 1$ the result is immediate. Suppose that $z \geq 2$, and consider an exact sequence

$$0 \longrightarrow \nabla(i,j) \xrightarrow{f} X \longrightarrow \operatorname{Coker} f \longrightarrow 0 \quad ,$$

with $(i,j) \in \Psi$ and $\operatorname{Coker} f$ in $\mathcal{F}(\nabla)$. Since $\mathcal{P}(X)$ holds, then $\mathcal{P}(\operatorname{Coker} f)$ also holds by the previous remark. By induction, $\operatorname{Coker} f$ must be a ∇-semisimple module. Consider now the pullback diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \nabla(i,j) & \longrightarrow & Y & \xrightarrow{h} & \nabla(k,l) & \longrightarrow & 0 \\ & & \Vert & & \downarrow{\scriptstyle \iota'} & & \downarrow{\scriptstyle \iota} & & \\ 0 & \longrightarrow & \nabla(i,j) & \xrightarrow{f} & X & \xrightarrow{\operatorname{coker} f} & \operatorname{Coker} f & \longrightarrow & 0 \end{array} \quad , \tag{5.9}$$

where ι is a split monic mapping the costandard module $\nabla(k,l)$ into the ∇-semisimple module $\operatorname{Coker} f$. We have that $\operatorname{Top} X \cong \operatorname{Top} \nabla(i,j) \oplus \operatorname{Top}(\operatorname{Coker} f)$, as the composition factors of X of type $L_{x,1}$ are exactly the summands of its top. According to Remark 1.3.10, $\operatorname{Top}(-)$ is a right exact functor. By applying $\operatorname{Top}(-)$ to (5.9), we get a new diagram where the bottom row remains exact, and consequently the top row also remains exact. Hence $\operatorname{Top} Y \ncong \operatorname{Top} \nabla(k,l)$. By Lemma 5.4.8, the top row of (5.9) splits. Since h is a split epic, there is a monic μ such that $h \circ \mu = 1_{\nabla(k,l)}$. Because h and ι are split maps, it is easy to see that $\iota' \circ \mu$ is a split monic. So $X \cong \nabla(k,l) \oplus X'$, for some module X' in $\operatorname{mod} B$. The module X' lies in $\mathcal{F}(\nabla)$ since this category is closed under quotients. In fact, $\mathcal{P}(X')$ holds by the observation in the beginning. By induction, the module X' must be ∇-semisimple. Therefore X is ∇-semisimple as well, which proves the first claim in the statement of the corollary.

Let now X' be a quotient of a ∇-semisimple module X. Then $\mathcal{P}(X)$ is true, which implies that $\mathcal{P}(X')$ holds. The first claim implies that X' is ∇-semisimple.

□

From Corollary 5.4.9, we conclude that ∇ is a cohereditary class in $\operatorname{mod}\Gamma$ (recall Definition 1.3.13). The next result follows by dualisation of the arguments and the reasoning in Subsection 3.2.2.

Proposition 5.4.10. *Using the previous notation (as in Subsection 5.4.1), assume that condition (D) is satisfied. Let X be a Γ-module in $\mathcal{F}(\nabla)$. Then $X/\operatorname{Rej}(X,\nabla)$ is the (unique) largest ∇-semisimple factor module of X. Furthermore,* $\operatorname{Rej}(X,\nabla)$ *lies in $\mathcal{F}(\nabla)$.*

Remark 5.4.11. Note that every semisimple module is cogenerated by the set ∇. Therefore $\operatorname{Top} X$ is a factor module of $X/\operatorname{Rej}(X,\nabla)$ for every X in $\operatorname{mod}\Gamma$.

We are now finally in position of proving Theorem 5.4.5.

Proof of Theorem 5.4.5. Assume that condition (D) holds for the WLUSQ algebra $(\Gamma,\Psi,\preceq)$. By Lemma 5.4.7, the Γ-modules $\nabla(i,1)$, $1 \leq i \leq n$, are all the simple costandard modules. In order to prove Theorem 5.4.5, one needs to show that $P_{i,1}$ lies in $\mathcal{F}(\nabla)$ for $1 \leq i \leq n$ (see Definition 5.4.1).

For this, consider the tilting module $T(i,l_i)$. The module $\nabla(i,l_i)$ is the largest ∇-semisimple factor module $T(i,l_i)/Y$ of $T(i,l_i)$ such that $Y \in \mathcal{F}(\nabla)$. By Proposition 5.4.10, it follows that $\nabla(i,l_i) \cong T(i,l_i)/\operatorname{Rej}(T(i,l_i),\nabla)$. According to Remark 5.4.11, the module $T(i,l_i)$ must have simple top isomorphic to $\operatorname{Top}\nabla(i,l_i) \cong L_{i,1}$. Since $\operatorname{Top} T(i,l_i) \cong \operatorname{Top} P_{i,1}$, there is an epic from $P_{i,1}$ to $T(i,l_i)$. By Theorem 4.3.13, there is a monic from $P_{i,1}$ to $T(i,l_i)$. This implies that $T(i,l_i) \cong P_{i,1}$, which concludes the proof of the theorem. □

Appendix A

The relationship between $\mathcal{R}(R_A)$ and $R_{A^{op}}$: an example

Consider the quiver

$$Q = \begin{array}{c} \beta \circlearrowleft \overset{1}{\circ} \circlearrowright \alpha \\ \Big\downarrow \gamma \\ \underset{2}{\circ} \circlearrowright \delta \end{array}$$

and define $A := KQ/ \operatorname{Rad}^2 KQ$. The projective indecomposable A-modules can be represented as

$$\begin{array}{ccccc} & & 1 & & \\ & {}^{\alpha}\swarrow & \mid & \searrow^{\beta} & \\ 1 & & 2 & & 1 \end{array} \quad , \quad \begin{array}{c} 2 \\ \mid \\ 2 \end{array} \; .$$

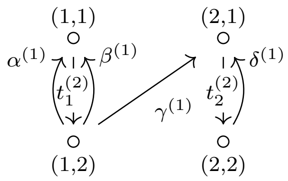
The injective indecomposable A-modules are

$$\begin{array}{ccc} 1 & & 1 \\ & {}_{\alpha}\searrow \;\; \swarrow_{\beta} & \\ & 1 & \end{array} \quad , \quad \begin{array}{ccc} 1 & & 2 \\ & \searrow \;\; \swarrow & \\ & 2 & \end{array} \; .$$

Note that A is such that the projective and injective indecomposable A-modules have all the same Loewy length and are rigid. Therefore, the statement in Theorem B holds for A. That is, the algebras $\mathcal{R}(R_A)$ and $(R_{A^{op}})^{op}$ are isomorphic. We shall check this directly.

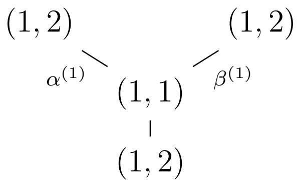
A.1 The algebra $\mathcal{R}(R_A)$

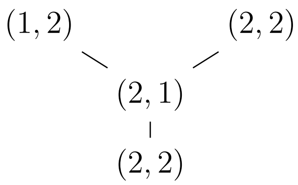
Using the labelling described in Section 2.2 the ADR algebra R_A of A is isomorphic to the algebra KQ'/I, where

$Q' =$ 

and I is the ideal generated by the relations $\alpha^{(1)}t_1^{(2)} = \beta^{(1)}t_1^{(2)} = 0$, $\gamma^{(1)}t_1^{(2)} = 0$ and $\delta^{(1)}t_2^{(2)} = 0$. Notice that this agrees with Proposition 3.3.2.

The indecomposable tilting R_A-modules are given

$Q_{1,2} = T(1,1) =$  ,

$Q_{2,2} = T(2,1) =$  ,

$T(1,2) = L_{1,2}$ and $T(2,2) = L_{2,2}$. Observe that this is in accordance with Lemma 3.4.1.

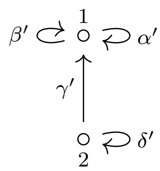
It is not difficult to see that the quiver associated to the Ringel dual

$$\mathcal{R}(R_A) = \mathrm{End}_{R_A}(T(1,1) \oplus T(1,2) \oplus T(2,1) \oplus T(2,2))^{op}$$

of R_A coincides with Q'. In fact, it is easy to check $\mathcal{R}(R_A)$ is isomorphic to KQ'/I', where I' is the ideal generated by the relations $t_1^{(2)}\alpha^{(1)} = t_1^{(2)}\beta^{(1)} = 0$, $t_2^{(2)}\delta^{(1)} = 0$, $t_2^{(2)}\gamma^{(1)} = 0$. To see that these are all the relations needed one can compare the dimensions of $\mathrm{End}_{R_A}(T)$ and KQ'/I'.

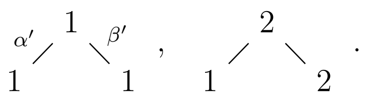
A.2 The algebra $(R_{A^{op}})^{op}$

Now we turn the attention to the algebra $(R_{A^{op}})^{op}$. The algebra A^{op} is isomorphic to $KQ_{(op)}/\operatorname{Rad}^2 KQ_{(op)}$, where

$Q_{(op)} =$ 

is obtained from Q by reversing the arrows.

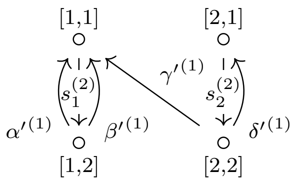
The projective indecomposble A^{op}-modules $P_i^{A^{op}}$ are given by



Using the labelling $[i, j]$ for the the projective indecomposable $R_{A^{op}}$-modules

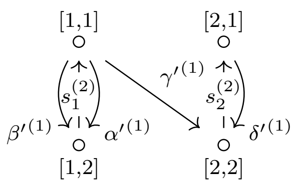
$$\operatorname{Hom}_{A^{op}}(G_{A^{op}}, P_i^{A^{op}}/\operatorname{Rad}^j P_i^{A^{op}})$$

it is not difficult to check that $R_{A^{op}}$ has a presentation $KQ'_{(op)}/I'$, with

$Q'_{(op)} =$ 

and I' the ideal generated by the relations $\alpha'^{(1)}s_1^{(2)} = \beta'^{(1)}s_1^{(2)} = 0$, $\delta'^{(1)}s_2^{(2)} = 0$, $\gamma'^{(1)}s_2^{(2)} = 0$.

So $(R_{A^{op}})^{op}$ is the quiver algebra of



,

bound by the relations $s_1^{(2)}\alpha'^{(1)} = s_1^{(2)}\beta'^{(1)} = 0$, $s_2^{(2)}\delta'^{(1)} = 0$, $s_2^{(2)}\gamma'^{(1)} = 0$.

A.3 Comparison

By comparing the quiver presentations of $\mathcal{R}(R_A)$ and $(R_{A^{op}})^{op}$, one concludes that these algebras are isomorphic, with the simple $(R_{A^{op}})^{op}$-modules labelled by $[i,j]$ corresponding to the $\mathcal{R}(R_A)$-modules with label $(i, 3-j)$. This observation agrees with the statement of Theorem B.

Appendix B

Natural ways of constructing hereditary preradicals

Recall the definitions of preradical, idempotent preradical and hereditary preradical given in Section 1.3 (see Definitions 1.3.1, 1.3.4 and 1.3.7).

Let A denote a C-algebra. Idempotent preradicals can always be written in the form $\mathrm{Tr}\,(\Theta, -)$ for some class of modules Θ in $\mathrm{Mod}\,A$ (consult Subsection 1.3.2). However, they do not always arise "in nature" in this form.

Since hereditary preradicals are idempotent, they can always be defined as the trace of a class of modules in $\mathrm{Mod}\,A$. We shall look at different natural ways of constructing hereditary preradicals.

For this, fix an object M in $\mathrm{Mod}\,A$. Let

$$\mathcal{R} : \mathrm{mod}\,A \longrightarrow \mathrm{mod}\,C$$

be a subfunctor of the functor

$$\mathrm{Hom}_A\,(M, -) : \mathrm{mod}\,A \longrightarrow \mathrm{mod}\,C.$$

The abelian group $\mathcal{R}(X) \in \mathrm{mod}\,C$ is a C-submodule of $\mathrm{Hom}_A\,(M, X)$ for all X in $\mathrm{mod}\,A$. Additionally, note that for X and Y in $\mathrm{mod}\,A$, we have $g \circ f \in \mathcal{R}\,(Y)$, for all $f \in \mathcal{R}\,(X)$ and all $g \in \mathrm{Hom}_A\,(X, Y)$. Finally, observe that $\mathcal{R}$ is an additive functor, that is $\mathcal{R}\,(f_1 + f_2) = \mathcal{R}\,(f_1) + \mathcal{R}\,(f_2)$, which is equivalent to saying that $\mathcal{R}$ preserves finite direct sums. Thus, a morphism f belongs to $\mathcal{R}\,(X)$, $X = \bigoplus_{i=1}^{n} X_i$, if and only if, $\pi_i \circ f$ is in $\mathcal{R}\,(X_i)$ for every i (here $\pi_i : X \longrightarrow X_i$ are the projection epics).

In what follows, whenever we say that $\mathcal{R}$ is a subfunctor of $\mathrm{Hom}_A\,(M, -)$, we mean that $\mathcal{R}$ is a subfunctor of

$$\mathrm{Hom}_A\,(M, -) : \mathrm{mod}\,A \longrightarrow \mathrm{mod}\,C.$$

Example B.1. Consider the subfunctor

$$\mathrm{Rad}_A(-,-) : \mathrm{mod}\, A^{op} \times \mathrm{mod}\, A \longrightarrow \mathrm{mod}\, C$$

of

$$\mathrm{Hom}_A\,(-,-) : \mathrm{mod}\, A^{op} \times \mathrm{mod}\, A \longrightarrow \mathrm{mod}\, C,$$

defined by

$$\mathrm{Rad}_A(X,Y) = \{f \in \mathrm{Hom}_A\,(X,Y) :\ 1_X - g \circ f \text{ is invertible, } \forall g \in \mathrm{Hom}_A\,(Y,X)\},$$

for X and Y in $\mathrm{mod}\, A$. This functor is called the *Jacobson radical* of $\mathrm{mod}\, A$. Obviously, for M in $\mathrm{mod}\, A$, $\mathrm{Rad}_A(M,-)$ is a subfunctor of $\mathrm{Hom}_A\,(M,-)$. Indeed, any subfunctor $I(-,-)$ of

$$\mathrm{Hom}_A\,(-,-) : \mathrm{mod}\, A^{op} \times \mathrm{mod}\, A \longrightarrow \mathrm{mod}\, C$$

gives rise to subfunctors $I(M,-)$ of $\mathrm{Hom}_A\,(M,-)$.

Example B.2. Let $f : M \longrightarrow M'$ be a morphism in $\mathrm{Mod}\, A$. The functor $\mathcal{R}_f := \mathrm{Im}\,(\mathrm{Hom}_A\,(f,-))$, which assigns to every X in $\mathrm{mod}\, A$ the set

$$\mathcal{R}_f(X) = \{g \circ f :\ g \in \mathrm{Hom}_A\,(M',X)\},$$

is a subfunctor of $\mathrm{Hom}_A\,(M,-)$.

Example B.3. Let $\mathcal{D}$ be a subcategory of $\mathrm{mod}\, A$, closed under (finite) direct sums. Consider the subfunctor

$$[\mathcal{D}](-,-) : \mathrm{mod}\, A^{op} \times \mathrm{mod}\, A \longrightarrow \mathrm{mod}\, C$$

of

$$\mathrm{Hom}_A\,(-,-) : \mathrm{mod}\, A^{op} \times \mathrm{mod}\, A \longrightarrow \mathrm{mod}\, C,$$

which assigns to every X, Y in $\mathrm{mod}\, A$ the set

$$[\mathcal{D}](X,Y) = \{f \in \mathrm{Hom}_A\,(X,Y) :\ f \text{ factors through a module in } \mathcal{D}\}.$$

This produces the subfunctors $[D](M,-)$ of $\mathrm{Hom}_A\,(M,-)$.

Any subfunctor $\mathcal{R}$ of $\mathrm{Hom}_A\,(M,-)$ gives rise to a functor

$$\tau_{\mathcal{R}} : \mathrm{mod}\, A \longrightarrow \mathrm{mod}\, A,$$

defined by

$$\tau_{\mathcal{R}}(X) = \sum_{f\colon f\in\mathcal{R}(X)} \operatorname{Im} f.$$

Observe that the module $\tau_{\mathcal{R}}(X)$ is generated by M.

Note that the functor $\tau_{\mathcal{R}}$ is a subfunctor of $1_{\operatorname{mod} A}$, or, in other words, $\tau_{\mathcal{R}}$ is a preradical in $\operatorname{mod} A$. Thus, $\tau_{\mathcal{R}}$ has several good properties, in particular, it is an additive functor, so it preserves finite direct sums. It is not difficult to see that $\tau_{\mathcal{R}}$ is actually a hereditary preradical in $\operatorname{mod} A$. In the next lemma we summarise some properties of these preradicals.

Lemma B.4. *Let $\mathcal{R}$ be a subfunctor of $\operatorname{Hom}_A(M,-)$, and let X be in $\operatorname{Mod} A$. Then:*

1. *$\tau_{\mathcal{R}}$ is an subfunctor of the identity functor which is left exact;*
2. *if Y is a submodule of X then $\tau_{\mathcal{R}}(Y) = \tau_{\mathcal{R}}(X) \cap Y$;*
3. *$\tau_{\mathcal{R}}(X)$ is generated by M;*
4. *$\tau_{\mathcal{R}}$ preserves finite direct sums, in particular, if $X = \bigoplus_{j=1}^n X_j$ then*

$$\tau_{\mathcal{R}}\left(\bigoplus_{j=1}^n X_j\right) = \bigoplus_{j=1}^n \tau_{\mathcal{R}}(X_j) = \bigoplus_{j=1}^n \tau_{\mathcal{R}}(X) \cap X_j.$$

5. *if $C = \bigoplus_{j=1}^n C_j$ and $\mathcal{R} = I(C,-)$ for some subfunctor $I(-,-)$ of*

$$\operatorname{Hom}_A(-,-) : \operatorname{mod} A^{op} \times \operatorname{mod} A \longrightarrow \operatorname{mod} C,$$

then $\tau_{\mathcal{R}}(X) = \sum_{j=1}^n \tau_{I(C_j,-)}(X)$.

Notation

All modules are left modules.

$\subset$	proper inclusion
$\subseteq$	inclusion
$\mathbb{Z}_{\geq 0}$	nonnegative integers
$\mathbb{Z}_{>0}$	positive integers
A, B	C-algebras (mostly Artin algebras), page 10
C	commutative artinian ring with unit
K	field
$(\Phi, \sqsubseteq)$	partial order
1_M	identity map on a set M (or identity functor if M is a category)

For a given C-algebra A

$\operatorname{Mod} A$	category of (left) A-modules
$\operatorname{mod} A$	category of the (left) A-modules which are finitely generated over C, page 8
A^{op}	opposite algebra of A
D	standard duality for an Artin algebra A, page 10
$\operatorname{proj} A$	category of the projective modules in $\operatorname{mod} A$
$\operatorname{gl.dim} A$	global dimension of A
$\operatorname{fin.dim} A$	finitistic dimension of A (Artin algebra), page 95

rep. dim A	representation dimension of A (Artin algebra), page 95

For a given A-module M (A C-algebra)

$\operatorname{Rad} M$	radical of M, page 10
$\operatorname{Soc} M$	socle of M, page 10
$\operatorname{Top} M$	top of M, page 10
$\operatorname{LL}(M)$	Loewy length of M in $\operatorname{mod} A$, page 30
$P_0(M)$	projective cover of $M \in \operatorname{mod} A$
$\Omega(M)$	kernel of the projective cover $P_0(M) \longrightarrow M$ of $M \in \operatorname{mod} A$
$\Omega^i(M)$	$\Omega(\Omega^{i-1}(M))$
$P_i(M)$	$P_0(\Omega^i(M))$
proj. dim M	projective dimension of M
$l(M)$	(Jordan–Hölder) length of M as a C-module

For a class Θ of modules in $\operatorname{mod} A$ (A C-algebra)

$\operatorname{add} \Theta$	category of modules isomorphic to summands of finite direct sums of modules in Θ
$\mathcal{F}(\Theta)$	category of the modules possessing a Θ-filtration, page 23

For given A-modules M and N (A C-algebra)

$[M : N]$	multiplicity of a simple module N in the composition series of $M \in \operatorname{mod} A$, page 10

For a given morphism $f : M \longrightarrow N$

$\operatorname{Im} f$	image of f
$\operatorname{Ker} f$	kernel of f
$\operatorname{Coker} f$	cokernel of f (i.e. $N/\operatorname{Im} f$)

$\operatorname{coker} f$	canonical epic $N \longrightarrow \operatorname{Coker} f$
$\ker f$	canonical monic $\operatorname{Ker} f \longrightarrow M$

For a given quasihereditary algebra $(B, \Phi, \sqsubseteq)$

$\Delta(i)$	standard module with label $i \in \Phi$, page 21
$\nabla(i)$	costandard module with label $i \in \Phi$, page 22
Δ	set of the all standard modules, page 22
∇	set of the all costandard modules, page 22
$(M : \Delta(i))$	multiplicity of $\Delta(i)$ in a Δ-filtration of $M \in \mathcal{F}(\Delta)$
$(M : \nabla(i))$	multiplicity of $\nabla(i)$ in a ∇-filtration of $M \in \mathcal{F}(\nabla)$

For a given RUSQ algebra $(B, \Phi, \sqsubseteq)$

$\Delta.\operatorname{ssl} M$	Δ-semisimple length of a module $M \in \mathcal{F}(\Delta)$, page 61
$\delta(-)$	$\operatorname{Tr}(\Delta, -)$, page 59
$\delta_m(-)$	$\delta \bullet \cdots \bullet \delta$ (m times), page 59

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